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有限深度設定下受激毛細重力波問題中微分運算子與級數間可交換性的一個證明

葉乃實

摘要

本論文旨在證明特定微分運算子與級數間的可交換性。此交換性，乃解決哈京氏接觸線及有限深度（液體）設定下、柱狀造波器外圍受激毛細重力波問題的根本要件。

A Proof of Interchangeability between differential operators and series involving the problem of forced Capillary-Gravity Waves in a Finite Depth Setting *†

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Abstract

The paper aims to prove interchangeability between specific differential operators and series, which is a fundamental fact constituting solvability of forced capillary-gravity waves problem outside a cylindrical wavemaker of finite depth under Hocking's edge condition.

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1. Introduction

The edge condition has been widely recongnized as an open question because of its complexity and uncertainty. In 1968, edge condition of fluid was first considered by Evans ([1], [2],) and Hocking [3] proposed a different model later in 1987. Both models were studied by various papers based upon different settings, including Miles ([4], [5]) and Shen *et al* ([8], [9], [10].) Originally Hocking's was considered physically more plausible than Evans's, but Miles ([4], [5]) argued that Hocking's setting was not practical in the case of a heaving cylinder with stick/slip edge condition. In 1995, his argument was supported by Ting and Perlin [6] through their study on edge condition, which used modern equipment to record edge condition of different types of fluid and consequently proposed a new edge condition. It has been regarded as the most practical model up to date.

According to the result found by Ting and Perlin, however, one may find that Hocking's edge condition is still valid concerning non-stick/-slip condition. Hence the problem on non-stick/-slip oscillating wavemaker under Hocking's edge condition needs further study. Constructing the exact solution of this problem in finite depth case is possible only if convergence of several series related to the problem is thoroughly studied. Similar research involving interchangeability of operators [12] may be compared to this assertion.

Here we focus our attention to the finite depth problem, present the related operators, and prove the convergence of these series.

2. The finite depth problem

The **governing equations** of the problem considered are as follows:

$$\mathcal{L}_2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0$$

$$\text{on } V = (a, \infty) \times (-1, 0) \text{ for some } a > 0; \quad (2.1)$$

$$-\omega^2\varphi + \varphi_z = \mathrm{T}\mathcal{L}_1\varphi_z = \mathrm{T}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2}\right)\varphi_z \text{ on } r > a,\ z = 0; \quad (2.2)$$

$$\varphi_z = 0 \quad \text{on} \quad z = -1, \quad (2.3)$$

$$\varphi_r = f(z) \quad \text{on} \quad r = a, \quad (2.4)$$

$$\varphi \to C_0\cosh(k_0(1+z))H_m^{(1)}(k_0 r) \text{ as } r \to \infty, \quad (2.5)$$

and the edge condition

$$\varphi_{rz} = \mathrm{i}\omega\delta\varphi_z \quad \text{at} \quad z = 0,\ r = a. \quad (2.6)$$

where

$$\mathcal{L}_2 = \mathcal{L}_1 + \frac{\partial^2}{\partial z^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}. \quad (2.7)$$

The original process of deriving these equations is shown in **Appendix,** and the equation (2.6) is **Hocking's Edge Condition.** Notice that φ is the potential function, T is the surface tension coefficient, ω is the angular frequency, m is the azimuthal number, δ is some real constant, f is an arbitrary smooth function, C_0 is a constant, k_0 is the positive real root of $\alpha(\mathrm{T}\alpha^2 + 1)\sinh\alpha - \omega^2\cosh\alpha = 0$, and $H_m^{(1)}(\cdot)$ is the Hankel's function of the first kind with order m. The following is an expansion theorem presented by Rhodes-Robinson [5], and proved by Yeh [8] when f is an arbitrary smooth function defined on $(-1,0)$:

Theorem 2.1. *An arbitrary smooth function $u(z)$ for $-1 < z < 0$ possesses a series expansion in the following form*

$$\begin{aligned} u(z) &= -4\pi \frac{k_0 A_0^*(\cosh k_0)(\cosh(k_0(1+z)))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0} \\ &\quad -4\pi \sum_{n=1}^{\infty} \frac{k_n A_n^*(\cos k_n)(\cos(k_n(1+z)))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n}, \end{aligned} \tag{2.8}$$

where $\pm k_0,\ \pm \mathrm{i}k_1,\ \pm \mathrm{i}k_2, \cdots\ \pm \mathrm{i}k_n \cdots$ are zeros of

$$\Delta(\alpha) = \alpha\left(\mathrm{T}\alpha^2+1\right)\sinh\alpha - \omega^2\cosh\alpha = 0, \tag{2.9}$$

$k_0 > 0,\ 0 < k_1 < k_2 < \cdots < k_n < \cdots,$

$$A_0^* = -\frac{1+\mathrm{T}k_0^2}{\pi(\cosh k_0)}\int_{-1}^{0} u(\xi)\cosh(k_0(1+\xi))d\xi + T\mu, \tag{2.10}$$

$$A_n^* = -\frac{1-\mathrm{T}k_n^2}{\pi(\cos k_n)}\int_{-1}^{0} u(\xi)\cos(k_n(1+\xi))d\xi + T\mu, \quad n=1,2,3,\cdots \tag{2.11}$$

and μ is an arbitrary parameter.

For simplicity, let

$$\beta_0 = 2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)\sinh 2k_0, \tag{2.12}$$

$$\beta_n = 2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)\sin 2k_n, \quad n=1,2,3,\cdots \tag{2.13}$$

and rewrite the expansion as

$$\begin{aligned} u(z) &= 2\left[\frac{2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+z)))\int_{-1}^{0} u\cosh(k_0(1+\xi))\mathrm{d}\xi}{\beta_0}\right. \\ &\quad \left.+\sum_{n=1}^{\infty}\frac{k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+z)))\int_{-1}^{0} u\cos(k_n(1+\xi))\mathrm{d}\xi}{\beta_n}\right] \\ &\quad -2\pi\mathrm{T}\mu\left[\frac{2k_0(\cosh k_0)(\cosh(k_0(1+z)))}{\beta_0}\right. \\ &\quad \left.+\sum_{n=1}^{\infty}\frac{k_n(\cos k_n)(\cos(k_n(1+z)))}{\beta_n}\right]. \end{aligned}$$

Then let

$$u(z) = 2u_1(z) - 2\pi\mathrm{T}\mu u_2(z), \qquad (2.14)$$

where

$$\begin{aligned} u_1(z) &= \frac{1}{\beta_0}\left[2k_0(1+\mathrm{T}k_0^2)(\cosh(k_0(1+z)))\int_{-1}^{0} u(\xi)\cosh(k_0(1+\xi))\mathrm{d}\xi\right] \\ &\quad +2\sum_{n=1}^{\infty}\frac{1}{\beta_n}\left[k_n(1-\mathrm{T}k_n^2)(\cos(k_n(1+z)))\int_{-1}^{0} u(\xi)\cos(k_n(1+\xi))\mathrm{d}\xi\right], \end{aligned} \qquad (2.15)$$

$$\begin{aligned} u_2(z) &= \frac{1}{\beta_0}\left[2k_0(\cosh k_0)(\cosh(k_0(1+z)))\right] \\ &\quad +2\sum_{n=1}^{\infty}\left[\frac{k_n}{\beta_n}(\cos k_n)(\cos(k_n(1+z)))\right]. \end{aligned} \qquad (2.16)$$

There is a zero term $-2\pi\mathrm{T}\mu u_2(z)$ in (2.14). However, the presence of zero term and the independent parameter μ have not been explained. It will become clear after we find the solution. We only note that μ will be determined by the edge condition.

The next theorem is the main result of this paper. By the conjecture of Yeh ([11], p.7), we obtain the series

$$\varphi(r,z) = \sum_{n=0}^{\infty} f_n(z)\gamma_n(r), \tag{2.17}$$

where

$$f(z) = \sum_{n=0}^{\infty} f_n(z), \tag{2.18}$$

$$f_0(z) = -4\pi \frac{k_0 A_0^*(\cosh k_0)\cosh(k_0(1+z))}{2k_0(1+\mathrm{T}k_0^2)+(1+3\mathrm{T}k_0^2)(\sinh 2k_0)}, \tag{2.19}$$

$$f_n(z) = -4\pi \frac{k_n A_n^*(\cos k_n)\cos(k_n(1+z))}{2k_n(1-\mathrm{T}k_n^2)+(1-3\mathrm{T}k_n^2)(\sin 2k_n)}, \quad n=1,2,\cdots, \tag{2.20}$$

which constitutes the possible solution of governing equations, and where A_j^*, k_j, $j=0,1,2,3,......$ are stated as in equations (2.9) to (2.11). Furthermore,

$$\gamma_0(r) = \frac{H_m^{(1)}(k_0 r)}{H_m^{(1)'}(k_0 a)},$$

$$\gamma_n(r) = \frac{K_m(k_n r)}{K_m'(k_n a)}, \quad n=1,2,3,....$$

where $H_m^{(1)}(\cdot)$ is Hankel's function of 1st kind of order m, $K_m(\cdot)$ is modified Bessel's function of 2nd kind of order m. When (2.17) is plugged into governing equations, the following series

$$\mathcal{L}_1 \sum_{n=0}^{\infty} f_n(z)\gamma_n(r), \tag{2.21}$$

$$\mathcal{L}_2 \sum_{n=0}^{\infty} f_n(z)\gamma_n(r), \tag{2.22}$$

$$\frac{\partial}{\partial r} \sum_{n=0}^{\infty} f_n(z)\gamma_n(r), \tag{2.23}$$

$$\frac{\partial}{\partial z} \sum_{n=0}^{\infty} f_n(z)\gamma_n(r), \text{ and} \tag{2.24}$$

$$\frac{\partial^2}{\partial z \partial r} \sum_{n=0}^{\infty} f_n(z)\gamma_n(r) \tag{2.25}$$

are encountered, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are expressed as in (2.7). So we have the next Theorem to claim that these differential operators can be taken inside of the series, which suggests that governing equations are solvable. Method of obtaining solution as well as related issue of the problem will be discussed in another paper.

Theorem 2.2. *The differential operators $\mathcal{L}_1$, $\mathcal{L}_2$, $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial z \partial r}$ can be taken into the series as shown in the above.*

Proof. All involving differential operators include $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial z}$, $\frac{\partial^2}{\partial r^2}$, $\frac{\partial^2}{\partial z^2}$, $\frac{\partial^2}{\partial z \partial r}$ and their linear combinations. Note that uniform convergence of the series $\sum_{n=0}^{\infty} f_n(z)\gamma_n(r)$ with respect to r and z garantees the interchangeability of the sum and the operators $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial z}$. Furthermore, that $\frac{\partial^2}{\partial r^2}$, $\frac{\partial^2}{\partial z^2}$ and $\frac{\partial^2}{\partial z \partial r}$ can be taken in to the sum only if the uniform convergence of $\sum_{n=0}^{\infty} f_n'(z)\gamma_n(r)$ and $\sum_{n=0}^{\infty} f_n(z)\gamma_n'(r)$ with respect to r and z may be established.

I). Let's look at the uniform convergence of series $\sum_{n=0}^{\infty} f_n(z)\gamma_n(r)$ first.

For sufficiently large n, we find that

$$\begin{aligned}
|\varphi_n(r,z)| &= |f_n(z)|\,|\gamma_n(r)| \\
&= \left| -4\pi \left[\frac{\frac{\cos k_n \cos(k_n(1+z))}{1-\mathrm{T}k_n^2}}{2+\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\sin 2k_n} \right] \times \left[-\left(\frac{1-\mathrm{T}k_n^2}{\pi\cos k_n}\right) \int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi + \mathrm{T}\mu \right] \right. \\
&\quad \left. \times \frac{K_m(k_n r)}{K_m'(k_n a)} \right| \\
&\leq \left| 4\pi \left\{ \frac{\frac{\cos k_n}{1-\mathrm{T}k_n^2}}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}} \times \left[\left(\frac{1-\mathrm{T}k_n^2}{\pi\cos k_n}\right)\|f\| - \mathrm{T}\mu \right] \right\} \times \frac{K_m(k_n r)}{K_m'(k_n a)} \right| \\
&= \left| \frac{4\|f\| - 4\pi\mathrm{T}\mu\left(\frac{\cos k_n}{1-\mathrm{T}k_n^2}\right)}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}} \right| \times \left| \frac{K_m(k_n r)}{K_m'(k_n a)} \right|, \qquad (2.26)
\end{aligned}$$

where

$$\|f\| = \max\{|f(z)|\ \ z\in[-1,0]\}, \qquad (2.27)$$

$$k_n = n\pi + \mathrm{Tan}^{-1}\left[\frac{\omega^2}{k_n(1-\mathrm{T}k_n^2)}\right], \quad \forall n\in\mathbb{N}, \qquad (2.28)$$

$$-1 \leq \cos k_n = \pm\frac{k_n(1-\mathrm{T}k_n^2)}{\sqrt{k_n^2(1-\mathrm{T}k_n^2)^2+\omega^4}} \leq 1, \qquad (2.29)$$

$$-1 \leq \sin k_n = \pm\frac{\omega^2}{\sqrt{k_n^2(1-\mathrm{T}k_n^2)^2+\omega^4}} \leq 1, \qquad (2.30)$$

$$-1 \leq \sin 2k_n \leq 1, \qquad (2.31)$$

$$|\cos(k_n(1+z))| \leq 1, \quad \forall z\in[-1,0],$$

and

$$\left|\int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi\right| \leq \|f\|. \qquad (2.32)$$

Since

$$\begin{aligned} K_m(r) &= \sqrt{\frac{\pi}{2r}} \frac{e^{-r}}{\left(m-\frac{1}{2}\right)!} \int_0^{\infty} e^{-t} t^{m-\frac{1}{2}} \left(1-\frac{t}{2r}\right)^{m-\frac{1}{2}} \mathrm{d}t, \\ K_m'(r) &= \frac{m}{r} K_m(r) - K_{m+1}(r), \quad r \geq a, \end{aligned}$$

we may find

$$\begin{aligned} K_m(r) &\sim \frac{e^{-r}}{\sqrt{\frac{2}{\pi} r}} \quad \text{when} \quad r >> m, \\ K_m'(r) &\sim \frac{e^{-r}}{\sqrt{\frac{2}{\pi} r}} \left(\frac{m}{r} - 1\right), \quad \text{for} \quad r >> m. \end{aligned}$$

Consequently,

$$\begin{aligned} K_m(k_n r) &\sim \sqrt{\frac{\pi}{2k_n r}} e^{-k_n r} \quad \text{and} \\ k_n K_m'(k_n r) &\sim \sqrt{\frac{\pi}{2k_n r}} e^{-k_n r} \left(\frac{m}{k_n r} - 1\right) \end{aligned}$$

for sufficiently large n, $r \geq a$. Hence

$$\gamma_n(r) \sim \left(\frac{k_n^2 a}{m - k_n a}\right) \sqrt{\frac{a}{r}} e^{-k_n(r-a)}. \tag{2.33}$$

By euqation (2.26) and the asymptotic expression in (2.33), we have

$$\begin{aligned} |\varphi_n(r,z)| &\leq \left| \frac{4\|f\| - \left(\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right)}{2 - \frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}} \right| \times \left| \frac{K_m(k_n r)}{K_m'(k_n a)} \right| \qquad (2.34) \\ &\sim \frac{|4\|f\|| + \left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2 - \frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|} \times \left| \frac{k_n^2 a}{k_n a - m} \right| \times \sqrt{\frac{a}{r}} \times e^{-k_n(r-a)}, \qquad (2.35) \end{aligned}$$

where $z \in [-1, 0]$, $k_n r >> m$ for sufficiently large n. Use Root Test, we find

$$\lim_{n\to\infty} \sqrt[n]{|\varphi_n(r,z)|} \leq \lim_{n\to\infty} \left[\frac{|4\|f\|| + \left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2 - \frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|} \right]^{\frac{1}{n}} \times \left| \frac{k_n a}{a - \frac{m}{k_n}} \right|^{\frac{1}{n}} \times \left(\frac{a}{r}\right)^{\frac{1}{2n}} \times e^{-\frac{k_n}{n}(r-a)}.$$

Then let's consider the following:

i).

$$\lim_{n\to\infty}\left[\frac{|4\,\|f\||+\left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right]^{\frac{1}{n}}=\lim_{n\to\infty}\exp\left[\frac{1}{n}\ln\left(\frac{|4\,\|f\||+\left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right)\right].$$

Since

$$\begin{aligned}\lim_{n\to\infty}\ln\left(\frac{|4\,\|f\||+\left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right) &= \lim_{n\to\infty}\left[\ln\left(|4\,\|f\||+\left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|\right)-\ln\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n\left(1-\mathrm{T}k_n^2\right)}\right|\right]\\ &= \ln\left(4\,\|f\|\right)-\ln 2=\ln\left(2\,\|f\|\right),\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\left[\frac{1}{n}\ln\left(\frac{|4\,\|f\||+\left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right)\right] &= \lim_{n\to\infty}\frac{1}{n}\cdot\ln\left(2\,\|f\|\right)=0\\ \Rightarrow\lim_{n\to\infty}\left[\frac{|4\,\|f\||+\left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right]^{\frac{1}{n}} &= 1. \qquad (2.36)\end{aligned}$$

ii).

$$\left|\frac{k_n a}{a-\frac{m}{k_n}}\right|^{\frac{1}{n}}=\exp\frac{1}{n}\ln\left(\frac{k_n a}{a-\frac{m}{k_n}}\right);$$

where

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{k_n a}{a-\frac{m}{k_n}}\right) &= \lim_{n\to\infty}\frac{1}{n}\left[\ln k_n a-\ln\left(a-\frac{m}{k_n}\right)\right]\\ &= \lim_{n\to\infty}\frac{\ln k_n a}{n}-\lim_{n\to\infty}\frac{1}{n}\ln\left(a-\frac{m}{k_n}\right).\end{aligned}$$

Note that

$$\lim_{n\to\infty}\frac{\ln k_n a}{n}=\lim_{n\to\infty}\frac{1}{n}\ln\left\{n\pi a+a\,\mathrm{Tan}^{-1}\left[\frac{\omega^2}{k_n\left(1-\mathrm{T}k_n^2\right)}\right]\right\}=0$$

for

$$\lim_{n\to\infty} \frac{\ln(n\pi a)}{n} = 0,$$

$$\lim_{n\to\infty} \mathrm{Tan}^{-1}\left[\frac{\omega^2}{k_n(1-\mathrm{T}k_n^2)}\right] = 0,$$

and $\frac{1}{n}\ln\left(a - \frac{m}{k_n}\right) \to 0$. Therefore,

$$\lim_{n\to\infty}\left|\frac{k_n a}{a - \frac{m}{k_n}}\right|^{\frac{1}{n}} = e^0 = 1. \tag{2.37}$$

iii).

$$\left(\frac{a}{r}\right)^{\frac{1}{2n}} = \exp\frac{1}{2n}(\ln a - \ln r), \quad r \geq a;$$

where

$$\lim_{n\to\infty}\frac{1}{2n}(\ln a - \ln r) = 0.$$

So

$$\lim_{n\to\infty}\left(\frac{a}{r}\right)^{\frac{1}{2n}} = e^0 = 1. \tag{2.38}$$

iv).

$$\begin{aligned}\lim_{n\to\infty}\frac{k_n}{n}(r-a) &= (r-a)\lim_{n\to\infty}\frac{1}{n}\left\{n\pi + \mathrm{Tan}^{-1}\left[\frac{\omega^2}{k_n(1-\mathrm{T}k_n^2)}\right]\right\} \\ &= (r-a)\lim_{n\to\infty}\pi + \lim_{n\to\infty}\frac{1}{n}\mathrm{Tan}^{-1}\left[\frac{\omega^2}{k_n(1-\mathrm{T}k_n^2)}\right] \\ &= \pi(r-a).\end{aligned}$$

Then we have

$$\lim_{n\to\infty} e^{-\frac{k_n}{n}(r-a)} = e^{-\pi(r-a)}. \tag{2.39}$$

Combining the results from (2.36) to (2.39), we conclude that

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{|\varphi_n(r,z)|} &\leq \lim_{n\to\infty}\left[\frac{|4\,\|f\|| + \left|\frac{4\pi \mathrm{T}\mu}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right]^{\frac{1}{n}} \times \left|\frac{k_n a}{a-\frac{m}{k_n}}\right|^{\frac{1}{n}} \times \left(\frac{a}{r}\right)^{\frac{1}{2n}} \times e^{-\frac{k_n}{n}(r-a)} \\ &\to 1\times 1\times 1\times e^{-\pi(r-a)} < 1.\end{aligned}$$

Thus we see that for $z\in[-1,0]$ and $r>a$, the series $\varphi(r,z)=\sum_{n=0}^{\infty} f_n(z)\gamma_n(r)$ converges uniformly, and the differential operators $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial r}$ are interchangeable with $\sum$ on $\{(r,z)|\ \ (r,z)\in[-1,0]\times(a,\infty)\}$.

II). Consider the uniform convergence of $\sum_{n=0}^{\infty} f_n'(z)\gamma_n(r)$:

$$\begin{aligned}\left|\frac{\partial}{\partial z}\varphi_n(r,z)\right| &= |f_n'(z)|\,|\gamma_n(r)| \\ &= \left|-4\pi\left[\frac{\frac{k_n\cos k_n \sin(k_n(1+z))}{1-\mathrm{T}k_n^2}}{2+\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\sin 2k_n}\right]\times\left[-\left(\frac{1-\mathrm{T}k_n^2}{\pi\cos k_n}\right)\int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi+\mathrm{T}\mu\right]\right. \\ &\quad \left.\times\frac{K_m(k_n r)}{K_m'(k_n a)}\right| \\ &\leq \left|4\pi\left\{\frac{\frac{k_n\cos k_n}{1-\mathrm{T}k_n^2}}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}}\times\left[\left(\frac{1-\mathrm{T}k_n^2}{\pi\cos k_n}\right)\|f\|-\mathrm{T}\mu\right]\right\}\times\frac{K_m(k_n r)}{K_m'(k_n a)}\right| \\ &= \left|\frac{4k_n\,\|f\| - 4\pi\mathrm{T}\mu k_n\left(\frac{\cos k_n}{1-\mathrm{T}k_n^2}\right)}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}}\right|\times\left|\frac{K_m(k_n r)}{K_m'(k_n a)}\right|. \end{aligned} \tag{2.40}$$

Following the identities from (2.27) to (2.32) and

$$|\sin(k_n(1+z))|\leq 1,\quad \forall z\in[-1,0],$$

we can see that

$$
\begin{aligned}
\left|\frac{\partial}{\partial z}\varphi_n(r,z)\right| &\leq \left|\frac{4k_n\,\|f\| - \left(\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right)}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}}\right| \times \left|\frac{K_m(k_n r)}{K_m'(k_n a)}\right| \\
&\sim \left[\frac{|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right] \left|\frac{k_n^2 a}{k_n a - m}\right| \sqrt{\frac{a}{r}} e^{-k_n(r-a)}, \quad (2.41)
\end{aligned}
$$

where $z \in [-1,0]$, $k_n r >> m$ for sufficiently large n. Use Root Test,

$$
\lim_{n\to\infty} \sqrt[n]{\left|\frac{\partial}{\partial z}\varphi_n(r,z)\right|} \leq \lim_{n\to\infty} \left[\frac{|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right]^{\frac{1}{n}} \times \left|\frac{k_n a}{a-\frac{m}{k_n}}\right|^{\frac{1}{n}} \times \left(\frac{a}{r}\right)^{\frac{1}{2n}} \times e^{-\frac{k_n}{n}(r-a)}.
$$

Look at the following:

i).

$$
\left[\frac{|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right]^{\frac{1}{n}} = \exp\frac{1}{n}\ln\left[\frac{|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right],
$$

and that

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\ln\left[\frac{|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right] \\
=\;& \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|\right) - \ln\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|\right] \\
=\;& \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(|4k_n\,\|f\||| + O\left(k_n^{-1}\right)\right) - \ln\left|2-O\left(k_n^{-1}\right)\right|\right] \\
=\;& \lim_{n\to\infty}\frac{1}{n}\left[\ln\left(|4k_n\,\|f\||\right) - \ln|2|\right] = 0,
\end{aligned}
$$

thus

$$
\lim_{n\to\infty}\left[\frac{|4k_n\,\|f\||| + \left|\frac{4\pi\mathrm{T}\mu k_n}{1-\mathrm{T}k_n^2}\right|}{\left|2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\right|}\right]^{\frac{1}{n}} = e^0 = 1. \quad (2.42)
$$

ii). Follow the results from (2.37) to (2.39) and (2.42),

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{\partial}{\partial z}\varphi_n(r,z)\right|} \leq 1 \times 1 \times 1 \times e^{-\pi(r-a)} < 1.$$

By Root Test, the series $\sum_{n=0}^{\infty} \frac{\partial}{\partial z}\varphi_n(r,z)$ converges uniformly for $z \in [-1,0]$ and $r > a$.

III). Consider the uniform convergence of $\sum_{n=0}^{\infty} f_n(z)\gamma_n'(r)$:

$$\begin{aligned}
\left|\frac{\partial}{\partial r}\varphi_n(r,z)\right| &= |f_n(z)|\,|\gamma_n'(r)| \\
&= \left|-4\pi \left[\frac{\frac{\cos k_n \cos(k_n(1+z))}{1-\mathrm{T}k_n^2}}{2+\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}\sin 2k_n}\right] \times \left[-\left(\frac{1-\mathrm{T}k_n^2}{\pi\cos k_n}\right)\int_{-1}^{0} f(\xi)\cos(k_n(1+\xi))\,\mathrm{d}\xi + \mathrm{T}\mu\right]\right. \\
&\quad \left. \times \frac{k_n K_m'(k_n r)}{K_m'(k_n a)}\right| \\
&\leq \left|4\pi\left\{\frac{\frac{\cos k_n}{1-\mathrm{T}k_n^2}}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}} \times \left[\left(\frac{1-\mathrm{T}k_n^2}{\pi\cos k_n}\right)\|f\| - \mathrm{T}\mu\right]\right\} \times \frac{k_n K_m'(k_n r)}{K_m'(k_n a)}\right| \\
&= \left|\frac{4\|f\| - 4\pi\mathrm{T}\mu\left(\frac{\cos k_n}{1-\mathrm{T}k_n^2}\right)}{2-\frac{1-3\mathrm{T}k_n^2}{k_n(1-\mathrm{T}k_n^2)}}\right| \times \left|\frac{k_n K_m'(k_n r)}{K_m'(k_n a)}\right|.
\end{aligned} \tag{2.43}$$

Since

$$\begin{aligned}
K_m(k_n r) &\sim \sqrt{\frac{\pi}{2k_n r}}e^{-k_n r} \quad \text{and} \\
k_n K_m'(k_n r) &\sim \sqrt{\frac{\pi}{2k_n r}}e^{-k_n r}\left(\frac{m}{k_n r} - 1\right)
\end{aligned}$$

for sufficiently large n, $r > a$. Hence

$$\gamma_n'(r) \sim k_n \left(\frac{r - \frac{m}{k_n}}{a - \frac{m}{k_n}}\right)\sqrt{\left(\frac{a}{r}\right)^3} e^{-k_n(r-a)}, \tag{2.44}$$

and

$$\left|\frac{\partial}{\partial r}\varphi_n(r,z)\right| \le \left|\frac{4\|f\| - \left(\frac{4\pi T\mu}{1-Tk_n^2}\right)}{2-\frac{1-3Tk_n^2}{k_n(1-Tk_n^2)}}\right| \times \left|\frac{k_n K'_m(k_n r)}{K'_m(k_n a)}\right|$$

$$\sim \left[\frac{|4k_n\|f\|| + \left|\frac{4\pi T\mu k_n}{1-Tk_n^2}\right|}{\left|2-\frac{1-3Tk_n^2}{k_n(1-Tk_n^2)}\right|}\right]\left|\left(\frac{r-\frac{m}{k_n}}{a-\frac{m}{k_n}}\right)\sqrt{\left(\frac{a}{r}\right)^3}\right| e^{-k_n(r-a)} \quad (2.45)$$

where $z \in [-1,0]$, $k_n r >> m$ for sufficiently large n. Use Root Test,

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{\partial}{\partial r}\varphi_n(r,z)\right|} \le \lim_{n\to\infty}\left\{\left[\frac{|4k_n\|f\|| + \left|\frac{4\pi T\mu k_n}{1-Tk_n^2}\right|}{\left|2-\frac{1-3Tk_n^2}{k_n(1-Tk_n^2)}\right|}\right]^{\frac{1}{n}} \times \left|\left(\frac{r-\frac{m}{k_n}}{a-\frac{m}{k_n}}\right)\left(\frac{a}{r}\right)^{\frac{3}{2}}\right|^{\frac{1}{n}} \times e^{-\frac{k_n}{n}(r-a)}\right\}.$$

Consider the following:

i).

$$\left(\frac{r-\frac{m}{k_n}}{a-\frac{m}{k_n}}\right)\left(\frac{a}{r}\right)^{\frac{3}{2}} = \left[\frac{r}{a} + O\left(k_n^{-1}\right)\right]\left(\frac{a}{r}\right)^{\frac{3}{2}}$$

$$= \left(\frac{a}{r}\right)^{\frac{1}{2}} + O\left(k_n^{-1}\right);$$

and then

$$\left|\left(\frac{r-\frac{m}{k_n}}{a-\frac{m}{k_n}}\right)\left(\frac{a}{r}\right)^{\frac{3}{2}}\right|^{\frac{1}{n}} = \exp\frac{1}{n}\ln\left[\left(\frac{a}{r}\right)^{\frac{1}{2}} + O\left(k_n^{-1}\right)\right],$$

so

$$\lim_{n\to\infty}\frac{1}{n}\ln\left[\left(\frac{a}{r}\right)^{\frac{1}{2}} + O\left(k_n^{-1}\right)\right] = 0,$$

$$\Rightarrow \lim_{n\to\infty}\left|\left(\frac{r-\frac{m}{k_n}}{a-\frac{m}{k_n}}\right)\left(\frac{a}{r}\right)^{\frac{3}{2}}\right|^{\frac{1}{n}} = e^0 = 1. \quad (2.46)$$

ii). Follow the results from (2.39), (2.44) and (2.48),

$$\lim_{n\to\infty}\sqrt[n]{\left|\frac{\partial}{\partial z}\varphi_n(r,z)\right|} \le 1\times 1\times 1\times e^{-\pi(r-a)} < 1.$$

By Root Test, the series $\sum_{n=0}^{\infty} \frac{\partial}{\partial r}\varphi_n(r,z)$ converges uniformly on $z \in [-1,0]$ and $r > a$.

Thus from the result in I)., $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial z}$ may be taken in to the series $\varphi(r,z) = \sum_{n=0}^{\infty} f_n(z)\gamma_n(r)$; results in II). and III). ensure that

$$\begin{aligned}
\frac{\partial^2 \varphi}{\partial r^2} &= \sum_{n=0}^{\infty} f_n(z)\gamma_n''(r), \\
\frac{\partial^2 \varphi}{\partial z^2} &= \sum_{n=0}^{\infty} f_n''(z)\gamma_n(r), \quad \text{and} \\
\frac{\partial^2 \varphi}{\partial z \partial r} &= \sum_{n=0}^{\infty} f_n'(z)\gamma_n'(r)
\end{aligned}$$

on $\{(r,z)|\ (r,z) \in [-1,0] \times (a,\infty)\}$. Therefore $\mathcal{L}_1$ and $\mathcal{L}_2$ can also be taken in to the sum $\sum_{n=0}^{\infty} f_n(z)\gamma_n(r)$. Now we completely proved the Theorem.

3. Conclusion

By using expansion theorem and a special assumption, we find that in order to construct the solution, the interchangeability between some differential operators and specific series should be verified first. Therefore the plausibility of proposing the theorem is asserted. We do not get into detail of solution finding process because that involves other technical details and is irrelevant to our proof. Construction of solution as well as possible application of the model shall be discussed in another paper.

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Appendix. Formulations

We shall consider capillary–gravity waves generated by a cylindrical wave-maker in an incompressible, inviscid fluid, and assume that the fluid motion is irrotational. Let us use a cylindrical coordinate system in which the z-axis is pointing vertically upwards, so that $z = 0,\ r > a$ is the undisturbed state of the fluid. The fluid region is exterior ($r > a$) to the wave maker. At equilibrium it is of uniform depth h. We may describe the fluid motion by a velocity potential $\Phi(r, \theta, z, t)$. The linearized equations governing the fluid motion are the following:

$$\nabla_3^2 \Phi = 0 \qquad \text{in the fluid region} \qquad V;$$

$$\left.\begin{aligned} \Phi_z &= Z_t, \\ \Phi_t + \mathrm{g}Z &= \mathrm{T}\nabla_2^2 Z \end{aligned}\right\} \quad \text{on} \quad z = 0,\ r > a;$$

where

$$V = \{(r, z)|\ r > a > 0, \text{ and } z \in (-h, 0)\};$$

∇_3^2 and ∇_2^2 denote three–dimensional and two–dimensional cylindrical Laplacians respectively, g is the gravitational constant, ρT is the surface tension constant, ρ is the fluid density, $\theta \in [0, 2\pi]$, and Z represents free surface of the fluid.

$$\Phi_r = f(z)e^{\mathrm{i}(\omega t \pm m\theta)} \qquad \text{on} \qquad r = a,$$

where ω is the angular frequency, m is the azimuthal number (i.e. the waves are generated asymmetrically,) and f is an arbitrary smooth function. The bottom condition is given by

$$\Phi_z \to 0 \qquad \text{on} \qquad z \to -1.$$

A radiation condition is prescribed as follows:

$$\Phi \to C_0 e^{k_0 z} H_m^{(1)}(k_0 r) e^{\mathrm{i}(\omega t \pm m\theta)} \text{ as } r \to \infty,$$

where $\pm k_0$ and $\pm \mathrm{i}k_n$, $n = 1, 2, 3, ...$; $k_0 > 0$, $k_1 < k_2 < k_3 < .....$ are the roots of equation

$$\Delta(\alpha) = \alpha(\mathrm{T}\alpha^2 + 1)\cosh\alpha - \omega^2 \sinh\alpha = 0,$$

$H_m^{(1)}(\cdot)$ is the Hankel's function of the first kind with order m, and C_0 is an unknown constant.

The edge condition prescribed for the problem here is **Hocking's edge condition,** and is given by

$$Z_t = \lambda Z_r, \quad \left(\lambda \equiv \frac{1}{\delta}\right) \qquad \text{at} \qquad r = a, \ z = 0,$$

where λ is a constant determined by experiment. Since the above equations are all linear, we may time–reduce and $\theta-$reduce the problem and assume

$$\begin{aligned} \Phi(r, \theta, z, t) &= \varphi_\infty(r, z) e^{\mathrm{i}(\omega t \pm m\theta)}, \\ Z(r, \theta, t) &= \overset{\wedge}{\zeta}(r) e^{\mathrm{i}(\omega t \pm m\theta)}. \end{aligned}$$

Now we measure r, z, Z and $\overset{\wedge}{\zeta}$ in units of 1, t in units of $\mathrm{g}^{-\frac{1}{2}}$; Φ, φ, T and λ in units of g; ω and f in units of $\mathrm{g}^{\frac{1}{2}}$. Then writing down the equations for the linearized and time-and-θ- reduced problem, we obtain the governing equations from (2.1) to (2.6).