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t-Tone Chromatic Numbers of Cycles of Length Less than Eight

Jun-Jie Pan *† Jing-Ru Wu‡

Abstract

A t-tone k-coloring of a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices u and v. The t-tone chromatic number of G, denoted $\tau_t(G)$, is the smallest positive integer k such that G has a t-tone k-coloring. For $t = 1$, $\tau_1(G) = \chi(G)$, is the chromatic number of a graph G.

For $t = 1$, the chromatic numbers of cycles are well-known. For $t = 2$, Bickle and Phillips [1] gave the 2-tone chromatic numbers of cycles. In this paper, we determine t-tone chromatic numbers of cycles of order n for $3 \leq n \leq 7$ and $t \geq 3$.

Keywords. t-tone chromatic number, cycle.

1 Introduction

Let $[k] = \{1, 2, ..., k\}$, t be a natural number, and $\binom{[k]}{t}$ denote the family of t-subsets of $[k]$. The notation $d(u, v)$ represents the distance between two vertices u and v of a graph G. A *t-tone k-coloring* of a graph $G = (V(G), E(G))$ is a function $f : V(G) \to \binom{[k]}{t}$ such that $|f(u) \cap f(v)| < d(u, v)$ for all distinct vertices u and v. The *t-tone chromatic number* of G, denoted $\tau_t(G)$, is the smallest positive integer k such that G has a t-tone k-coloring. For $t = 1$, $\tau_1(G) = \chi(G)$, is the chromatic number of a graph G. The motivation of this topic is mentioned in the paper [2].

*Department of Mathematics, Fu Jen Catholic University, New Taipei City 24205, Taiwan.
†E-mail: jackpan@math.fju.edu.tw.
‡Department of Mathematics, Fu Jen Catholic University, New Taipei City 24205, Taiwan.

A *cycle* C_n of n vertices is a graph with vertex set $V(C_n) = \{v_i : i \in [n]\}$ and edge set $E(C_n) = \{v_iv_{i+1} : i \in [n-1]\} \cup \{v_nv_1\}$. In this paper, we consider this problem on cycles. For $t = 1$, the chromatic numbers of cycles are well-known. For $t = 2$, Bickle and Phillips [1] gave the 2-tone chromatic numbers of cycles. We determine t-tone chromatic numbers of cycles of order n for $3 \leq n \leq 7$ and $t \geq 3$.

2 Previous Results

We list some tools that help us build our results in this section.

Proposition 1. [1] *$\tau_t(K_n) = tn$ for natural numbers t and n.*

Proposition 2.

$$\tau_1(C_n) = \begin{cases} 2 & \textit{if } n \textit{ is even,} \\ 3 & \textit{otherwise.} \end{cases}$$

Proposition 3. [1]

$$\tau_2(C_n) = \begin{cases} 6 & \textit{if } n = 3, 4, 7, \\ 5 & \textit{otherwise.} \end{cases}$$

Proposition 4. [1] *Let P_n be the path of order n. Then*

$$\tau_t(P_n) = \sum_{i=0}^{n-1} \max\left\{0, t - \binom{i}{2}\right\}.$$

Proposition 5. [1] *If H is a subgraph of a graph G, then $\tau_t(H) \leq \tau_t(G)$.*

3 Main Results

In this section, we will determine t-tone chromatic numbers of cycles of order n for $3 \leq n \leq 7$ and $t \geq 3$. First of all, we get a natural lower bound for t-tone chromatic numbers of cycles in the following proposition.

Proposition 6.

$$\tau_t(C_n) \geq \sum_{i=0}^{m-1} \max\left\{0, t - \binom{i}{2}\right\} \quad \textit{for } m \leq n.$$

Proof. By Proposition 4, $\tau_t(P_m) = \sum_{i=0}^{m-1} \max\left\{0, t - \binom{i}{2}\right\}$. Since P_m is a subgraph of C_n for $m \le n$, by Proposition 5 this proposition holds. $\square$

Next, we get lower bounds for t-tone chromatic numbers of cycles C_n with small n and sufficiently large t by using the Inclusion-Exclusion Principle in Lemma 1.

Lemma 1. *For every natural number* t,

$$\tau_t(C_n) \ge \begin{cases} 4t-2 & \text{if } n=4, \\ 5t-5 & \text{if } n=5, \\ 6t-12 & \text{if } n=6, \\ 7t-21 & \text{if } n=7. \end{cases}$$

Proof. Suppose that $V(C_n) = \{v_i : i \in [n]\}$ and f is a t-tone k-coloring of C_n for $4 \le n \le 7$. Then $d(v_i, v_j) \le 3$ for $i \ne j$. Let $f(v_i) = A_i$ for all $i \in [n]$. Then by the Pigeonhole Principle and $|f(u) \cap f(v)| < d(u,v) = 1$ for all distinct adjacent vertices u and v, $|\cap_{i \in I} A_i| = 0$ for $|I| \ge 4$. Thus by the Inclusion-Exclusion Principle, we have

$$\begin{aligned} k &= \left| \bigcup_{i=1}^{n} A_i \right| \\ &= \sum_{i=1}^{n} |A_i| - \sum_{\substack{1 \le i < j \le n \\ d(v_i,v_j)=1}} |A_i \cap A_j| - \sum_{\substack{1 \le i < j \le n \\ d(v_i,v_j)=2}} |A_i \cap A_j| \\ &\quad - \sum_{\substack{1 \le i < j \le n \\ d(v_i,v_j)=3}} |A_i \cap A_j| + \sum_{\emptyset \ne I \subseteq [n],\ |I| \ge 3} (-1)^{|I|+1} \left| \bigcap_{i \in I} A_i \right| \\ &\ge nt - \sum_{\substack{1 \le i < j \le n \\ d(v_i,v_j)=2}} |A_i \cap A_j| - \sum_{\substack{1 \le i < j \le n \\ d(v_i,v_j)=3}} |A_i \cap A_j| \end{aligned}$$

Let $b_m = \sum_{\substack{1 \le i < j \le n \\ d(v_i,v_j)=m}} |A_i \cap A_j|$. For $n=4$, $b_2 \le 2$ and $b_3 = 0$. For $n=5$, $b_2 \le 5$ and $b_3 = 0$. For $n=6$, $b_2 \le 6$ and $b_3 \le 3 \cdot 2$. For $n=7$, $b_2 \le 7$ and $b_3 \le 7 \cdot 2$. Therefore, this lemma holds. $\square$

Recall that the *Wiener index* of a connected graph $G = (V, E)$, written $W(G) = \sum_{u,v \in V} d(u,v)$, is the sum of the distances of all pairs of vertices of G.

Lemma 2. *If $t \geq \lfloor n^2/4 \rfloor - n + 1$, then $\tau_t(C_n) \leq tn - W(C_n) + \binom{n}{2}$.*

Proof. Observe that for every vertex $v \in V(C_n)$, $\sum_{u \neq v}[d(u,v) - 1] = \lfloor n^2/4 \rfloor - n + 1$. It suffices to construct a t-tone $tn - W(C_n) + \binom{n}{2}$-coloring f of C_n. Since $t \geq \lfloor n^2/4 \rfloor - n + 1$, the set $[tn - W(G) + \binom{n}{2}]$ is arbitrarily partitioned into n disjoint subsets of specific sizes. Indeed, let $[tn - W(C_n) + \binom{n}{2}] = \bigcup_{i=1}^{n} I_{ii}$, where $|I_{11}| = t$, $|I_{ii}| = t - \sum_{j=1}^{i-1}[d(v_j, v_i) - 1]$ for $2 \leq i \leq n$, and $I_{ii} \cap I_{jj} = \emptyset$ for $i \neq j$. Second, since $t \geq \lfloor n^2/4 \rfloor - n + 1$, $|I_{11}| = t$, and $|I_{ii}| = t - \sum_{j=1}^{i-1}[d(v_j, v_i) - 1]$ for $2 \leq i \leq n$, we can find $n - i$ arbitrarily disjoint subsets of I_{ii} for $i \in [n]$. That is, let $I_{ii} \supseteq \bigcup_{j=i+1}^{n} I_{ij}$ with $|I_{ij}| = d(v_i, v_j) - 1$ and $I_{ij} \cap I_{ik} = \emptyset$ for $j \neq k$. Third, define $f(v_i) = \bigcup_{j=1}^{i} I_{ji}$ for $i \in [n]$. Then $|f(v_1)| = t$ and $|f(v_i)| = \sum_{j=1}^{i} |I_{ji}| = \sum_{j=1}^{i-1}[d(v_j, v_i) - 1] + |I_{ii}| = t$ for $2 \leq i \leq n$. For $i < j$, since $I_{ki} \cap I_{kj} = \emptyset$ for $k \in [i-1]$ and $I_{ii} \supseteq I_{ij}$, we have $|f(v_i) \cap f(v_j)| = |I_{ij}| = d(v_i, v_j) - 1 < d(v_i, v_j)$. Therefore, we have a t-tone $tn - W(C_n) + \binom{n}{2}$-coloring of C_n. □

For the above lemma, we have a stronger result for a connected graph in [3].

From now on, we will give t-tone chromatic numbers of cycles of order n for $3 \leq n \leq 7$ in the following theorems.

Theorem 1. *$\tau_t(C_4) = 4t - 2$ for every natural number t.*

Proof. Since $W(C_4) = 8$, by Lemma 2, $\tau_t(C_4) \leq 4t - 2$. Together with Lemma 1, this theorem holds. □

Theorem 2.

$$\tau_t(C_5) = \begin{cases} 3 & \textit{if } t = 1, \\ 5t - 5 & \textit{if } t \geq 2. \end{cases}$$

Proof. Since $W(C_5) = 15$, by Lemma 2, $\tau_t(C_4) \leq 5t - 5$ for $t \geq 2$. Together with Lemma 1 and Proposition 2, this theorem holds. □

Proposition 7. $\tau_3(C_6) = 8$.

Proof. By Proposition 6, $8 = \tau_3(P_3) \le \tau_3(C_6)$.

On the other hand, let the function $f : V(C_6) \to [8]$ with $f(v_1) = \{1,2,3\}, f(v_2) = \{4,5,6\}, f(v_3) = \{1,7,8\}, f(v_4) = \{2,3,4\}, f(v_5) = \{1,5,6\}, f(v_6) = \{4,7,8\}$. Then f is a 3-tone 8-coloring of C_6 implies that $\tau_3(C_6) \le 8$. Hence,$\tau_3(C_6) = 8$ □

Theorem 3.

$$\tau_t(C_6) = \begin{cases} 2 & \text{if } t = 1, \\ 5 & \text{if } t = 2, \\ 8 & \text{if } t = 3, \\ 6t - 12 & \text{if } t \ge 4. \end{cases}$$

Proof. Since $W(C_6) = 27$, by Lemma 2, $\tau_t(C_6) \le 6t - 12$ for $t \ge 4$. By Lemma 1, $\tau_t(C_6) = 6t - 12$ for $t \ge 4$. Together with Propositions 2-3 and Proposition 7, this theorem holds. □

Lemma 3. *Suppose that f is a t-tone $(4t-4)$-coloring of C_7 for $t \ge 3$, then $|f(u) \cap f(v)| = 2$ for all $d(u,v) = 3$ and $|f(u) \cap f(v)| = 1$ for all $d(u,v) = 2$.*

Proof. Let $v_0, v_3 \in V(C_7)$ with $d(v_0, v_3) = 3$. Then $v_0v_1v_2v_3$ be a path P_4. Sine f is a t-tone $(4t-4)$-coloring and $\tau_4(P_4) = 4t-4$, $4t-4 = |\cup_{i=0}^{3} f(v_i)|$. Thus by the Inclusion-Exclusion Principle, $4t - 4 = |\cup_{i=0}^{3} f(v_i)| = 4t - |f(v_0) \cap f(v_2)| - |f(v_0) \cap f(v_3)| - |f(v_1) \cap f(v_3)|$ implies that $|f(v_0) \cap f(v_3)| = 2, |f(v_0) \cap f(v_2)| = 1, |f(v_1) \cap f(v_2)| = 1$ because $|f(v_0) \cap f(v_2)| \le 1$, $|f(v_1) \cap f(v_3)| \le 1$, and $|f(v_0) \cap f(v_3)| \le 2$. Hence, this lemma holds. □

Proposition 8. $\tau_3(C_7) = 9$.

Proof. Suppose that C_7 has a 3-tone 8-coloring. Then by Lemma 3, we can assume that $f(v_1) = \{1,2,3\}, f(v_2) = \{4,5,6\}, f(v_3) = \{1,7,8\}$, and $f(v_4) = \{2,3,4\}$.

Since $d(v_1, v_5) = 3$ and $d(v_2, v_5) = 3$, by Lemma 3 $|f(v_1) \cap f(v_5)| = 2$ and $|f(v_2) \cap f(v_5)| = 2$. Since $|f(v_1) \cap f(v_2)| = 0$, $|f(v_1) \cap f(v_5)| + |f(v_2) \cap f(v_5)| = 4 \le$

$|f(v_5)| = 3$, a contradiction. Thus C_7 does not have a 3-tone 8-coloring. Therefore $\tau_3(C_7) \geq 9$.

On the other hand, let $f(v_1) = \{1,2,3\}, f(v_2) = \{4,5,6\}, f(v_3) = \{1,7,8\}$, $f(v_4) = \{3,4,9\}, f(v_5) = \{1,5,6\}, f(v_6) = \{2,4,7\}$, and $f(v_7) = \{5,8,9\}$. Then f is 3-tone 9-coloring implies that $\tau_3(C_7) \leq 9$. Hence $\tau_3(C_7) = 9$. □

Proposition 9. $\tau_4(C_7) = 13$.

Proof. Suppose that C_7 has a 4-tone 12-coloring. Then by Lemma 3, we can assume that $f(v_1) = \{1,2,3,4\}, f(v_2) = \{5,6,7,8\}, f(v_3) = \{1,9,10,11\}$, and $f(v_4) = \{2,3,5,12\}$.

Since $5 \in f(v_4)$ and $d(v_2, v_5) = 3$, by Lemma 3, assume that $\{6,7\} \subseteq f(v_5)$. Since $\{2,3\} \subseteq f(v_4)$ and $d(v_1, v_5) = 3$, by Lemma 3, $f(v_5) = \{1,4,6,7\}$.

Since $\{6,7\} \subseteq f(v_5)$ and $d(v_2, v_6) = 3$, $\{5,8\} \subseteq f(v_6)$. Since $1 \in f(v_5)$ and $d(v_3, v_6) = 3$, we can assume that $f(v_6) = \{5,8,9,10\}$.

Since $\{9,10\} \subseteq f(v_6)$ and $d(v_3, v_7) = 3$, $\{1,11\} \subseteq f(v_7)$. Since $d(v_1, v_7) = 1$, $1 \notin f(v_7)$, a contradiction. Thus, $\tau_4(C_7) \geq 13$.

On the other hand, let $f(v_1) = \{1,2,3,4\}, f(v_2) = \{5,6,7,8\}, f(v_3) = \{1,9,10,11\}$, $f(v_4) = \{2,3,5,12\}, f(v_5) = \{1,4,6,7\}, f(v_6) = \{5,8,9,10\}$, and $f(v_7) = \{6,11,12,13\}$. Then f is 4-tone 13-coloring implies that $\tau_3(C_7) \leq 13$. Hence $\tau_4(C_7) = 13$. □

Proposition 10. $\tau_5(C_7) = 17$.

Proof. Suppose that C_7 has a 5-tone 16-coloring. Then by Lemma 3, we am assume that $f(v_1) = \{1,2,3,4,5\}$, $f(v_2) = \{6,7,8,9,10\}$, $f(v_3) = \{1,11,12,13,14\}$, and $f(v_4) = \{2,3,6,15,16\}$.

Suppose that $1 \in f(v_5)$. Since $d(v_5, v_1) = 3$, and $\{2,3\} \subseteq f(v_4)$, by Lemma 3, assume that $\{1,4\} \subseteq f(v_5)$. Since $6 \in f(v_4)$ and $f(v_2) = \{6,7,8,9,10\}$, by Lemma 3, assume that $\{7,8\} \subseteq f(v_5)$ and so $f(v_5) = \{1,4,7,8,a\}$ for some $a \in [16]$. Together with $d(v_5, v_3) = 2$ and $d(v_5, v_4) = 1$, by Lemma 3, we have $a \notin [16]$, a contradiction.

Now, $1 \notin f(v_5)$. Since $d(v_5, v_1) = 3$ and $d(v_5, v_4) = 1$, by Lemma 3, $\{4,5\} \subseteq f(v_5)$. Since $6 \in f(v_2)$, by Lemma 3, assume that $\{7,8\} \subseteq f(v_5)$. Since $d(v_5, v_3) = 1$, by Lemma 3, assume that $f(v_5) = \{4,5,7,8,11\}$.

Suppose that $6 \in f(v_6)$. Since $d(v_6, v_2) = 3$, and $d(v_6, v_5) = 1$, by Lemma 3, assume that $\{6,9\} \subseteq f(v_6)$. Since $d(v_6, v_3) = 3$ and $d(v_6, v_5) = 1$, by Lemma 3, assume that $f(v_6) = \{6,9,1,12,b\}$ or $\{6,9,12,13,b\}$ for some $b \in [16]$. Since $6 \in f(v_4)$, $b \notin [16]$, a contradiction. Thus, $6 \notin f(v_6)$. Since $d(v_6, v_2) = 3$, and $d(v_6, v_5) = 1$, by Lemma 3, we can assume that $\{9,10\} \subseteq f(v_6)$.

Since $d(v_6, v_3) = 3$ and $d(v_6, v_5) = 1$, by Lemma 3, assume that $\{1,12\} \subseteq f(v_6)$ or $\{12,13\} \subseteq f(v_6)$. For $\{1,12\} \subseteq f(v_6)$, since $d(v_6, v_1) = 2$ and $d(v_6, v_4) = 2$, by Lemma 3, assume that $f(v_6) = \{1,9,10,12,15\}$. Thus, we have two cases for $f(v_6)$.

Case 1. $f(v_6) = \{1,9,10,12,15\}$. In this case, since $d(v_7, v_6) = 1$, and $d(v_7, v_3) = 3$, by Lemma 3, assume that $f(v_7) = \{11,13,c,d,e\}$ or $\{13,14,c,d,e\}$ for some $c,d,e \in [16]$. Suppose that $f(v_7) = \{11,13,c,d,e\}$. Since $d(v_7, v_1) = 1$ and $d(v_7, v_5) = 2$, by Lemma 3, we have $\{c,d,e\} \subseteq \{6,16\}$, a contradiction. For $f(v_7) = \{13,14,c,d,e\}$, since $d(v_7, v_4) = 3$, by Lemma 3, $f(v_7) = \{13,14,6,16,e\}$. Since $d(v_7, v_2) = 2$ and $d(v_7, v_5) = 2$, by Lemma 3, we have $e \notin [16]$, a contradiction.

Case 2. $\{9,10,12,13\} \subseteq f(v_6)$. In this case, since $d(v_7, v_1 = 1)$, $f(v_7, v_6) = 1$, and $d(v_7, v_3) = 3$, by Lemma 3, $f(v_7) = \{11,14,c,d,e\}$ for some $c,d,e \in [16]$. Since $11 \in f(v_5)$, by Lemma 3, $\{c,d,e\} = \{6,15,16\}$ contrary to the inequality $3 = |f(v_7) \cap f(v_4)| < d(v_7, v_4) = 2$. Therefore, $\tau_5(C_7) \geq 17$.

On the other hand, let the function $f : V(C_7) \to [17]$ with $f(v_1) = \{1,2,3,4,5\}$, $f(v_2) = \{6,7,8,9,10\}$, $f(v_3) = \{1,11,12,13,14\}$, $f(v_4) = \{2,3,6,15,16\}$, $f(v_5) = \{4,5,7,8,11\}$, $f(v_6) = \{2,9,10,12,13\}$, and $f(v_7) = \{11,14,15,16,17\}$. Then f is a 5-tone 17-coloring of C_7

implies that $\tau_5(C_7) \leq 17$. Hence,$\tau_5(C_7) = 17$. □

Theorem 4.

$$\tau_t(C_7) = \begin{cases} 3 & if\, t = 1, \\ 6 & if\, t = 2, \\ 9 & if\, t = 3, \\ 13 & if\, t = 4, \\ 17 & if\, t = 5, \\ 7t - 21 & if\, t \geq 6. \end{cases}$$

Proof. Since $W(C_7) = 42$, by Lemma 2, $\tau_t(C_7) \leq 7t - 21$ for $t \geq 6$. Together with Lemma 1 and Propositions 3, 8-10, this theorem holds. □

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