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FU JEN UNIVERSITY

TAIPEI, TAIWAN, REPUBLIC OF CHINA

COHOMOLOGY OF COCOMMUTATIVE HOPF ALGEBRAS

WEN-HSIUNG LIN

§1. INTRODUCTION

In this paper we shall construct another version of the May spectral sequence for cocommutative Hopf algebras. The purpose of the study of this kind is to find an effective method for computing the cohomology of such algebras.

A major problem in homological algebra is the effective calculation of $\mathrm{Ext}_A(K, K)$ for a graded Hopf algebra A over a field K. In topology, the problem comes up in evaluating the E_2-term of the Adams spectral sequence. In the latter case the algebras we are interested in turns out to be cocommutative Hopf algebras over Z_p, the ring of integers modulo a prime number p.

In recent years the most successful method to attack this problem has been the May spectral sequence. Let A be a Hopf algebra over a field K and let $E^0(A)$ be the associated graded algebra corresponding to the augmentation filtration $F_p(A)$ of A. J. P. May[1] establishes a spectral sequence $\{E_r\}$ with $E_2=H^*(E^0(A))$ which converges to an algebra associated to $H^*(A)$. Here $H^*(X)=\mathrm{Ext}_X(K, K)$ denotes the cohomology of a graded algebra X (over K). Since E^0A is primitively generated, by a theorem due to Milnor and Moore[2], it is isomorphic to the universal enveloping algebra of the Lie algebra or the restricted Lie algebra (depending on whether K is of characteristic 0 or of characteristic a prime number) of its primitive elements. In his dissertation, May also gives a canonical resolution of K over such a universal enveloping algebra. Since $H^*(E^0A)$ is more manageable and much smaller than the cobar construction $\bar{c}(A^*)$ it is easier to start from $H^*(E^0(A))$, the E_2-term of the May spectral sequence, to calculate $H^*(A)$.

In this paper we develope a technique similar to May's and establish a spectral sequence for cocommutative Hopf algebras

parallel to May spectral sequence. The main theorem will be stated at the end of the next section. In §3 we apply our method to derive May's resolution for restricted Lie algebra in a neat and elegant way. This is the only application of our method in this paper. Our main goal is, of course, to apply this method to the computations of the cohomology of the Steenrod algebra. We hope to come to this problem in later papers.

§2. CONSTRUCTION OF A SPECTRAL SEQUENCE

In this paper we only consider a mod 2 case, i. e. the ground field will be Z_2. Our construction can be applied to any locally finite cocommutative Hopf algebra, but we shall just do this for the mod 2 Steenrod algebra as an illustration. At the end of this section we summarize our result for all cocommutative Hopf algebras.

So let A denote mod 2 Steenrod algebra and let A* be the dual Hopf algebra. Thus $A^*=Z_2[\xi_1, \xi_2, \dots]$

J. P. May considers the augmentation filtration F_pA of A to establish his spectral sequence. Instead of this we consider the augmentation filtration $F_p(A^*)$ of A*. That is,

$F_pA^*=A^*$ if $p\geqslant 0$

$F_{-1}A^*=I(A^*)$

$F_{-p}A^*=\mathrm{Im}\ (\underbrace{I(A^*)\otimes\dots\otimes I(A^*)}_{p}\xrightarrow{\text{multiplication}}I(A^*))$ if $p>1$.

Let $E^0(A^*)$ be the associated graded algebra i. e.

$$E^0_{p,q}(A^*)=(F_pA^*/F_{p-1}A^*)_{p+q}.$$

By *Theorem 7.13* of Ref. (2), $E^0A^*\simeq A^*$, since A* is a commutative Hopf algebra. So $E^0A^*\simeq A^*=Z_2[\xi_1, \xi_2, \dots]$. By *Theorem 7.4* of Ref. (2), E^0A^* is primitively generated. So in E^0A^* we have $\Delta(\xi_k)=\xi_k\otimes 1+1\otimes\xi_k$ all k. Let E^0A be the dual Hopf algebra of E^0A^*. The reason we write this dual Hopf algebra as E^0A is the following: If we denote by $F_p(A)$ the filtration of A dual to $F_p(A^*)$ then E^0A is precisely the associated graded algebra. Now E^0A is a divided polynomial algebra with usual coproduct. More precisely, if $\left\{r\left(\xi_1^{i_1}\dots\xi_n^{i_n}\right)\right\}$ is the Z_2-basis of E^0A dual to the nonomial basis $\left\{\left(\xi_1^{i_1}\dots\xi_n^{i_n}\right)\right\}$

of E^0A^*, then the multiplication of E^0A is given by

$$\gamma\left(\xi_1^{i_1}\dots\xi_n^{i_n}\right)\gamma\left(\xi_1^{j_1}\dots\xi_n^{j_n}\right)=\binom{i_1+j_1}{i_1}\dots\binom{i_n+j_n}{i_n}\gamma\left(\xi_1^{i_1+j_1}\dots\xi_n^{i_n+j_n}\right).$$

and the coproduct is given by

$$\Delta\left[\gamma\left(\xi_1^{i_1}\dots\xi_n^{i_n}\right)\right]=\sum_{\nu_1=0}^{i_1}\dots\sum_{\nu_n=0}^{i_n}\gamma\left(\xi_1^{\nu_1}\dots\xi_n^{\nu_n}\right)\otimes\gamma\left(\xi_1^{i_1-\nu_1}\dots\xi_n^{i_n-\nu_n}\right).$$

Let $a_j^i=\gamma(\xi_j^{2^i})$. Then it is easy to see that, as an algebra, E^0A is just an exterior algebra $E[a_j^i]$. We already know that the cohomology $H^*(E^0A)$ of E^0A is a polynomial algebra $Z_2[R_j^i]$ with $\dim R_j^i=\dim a_j^i$. Here the homological degree of R_j^i in $H^*(E^0A)$ is 1 and $\dim R_j^i$ is the t-degree in $H^{s,t}(E^0A)=\mathrm{Ext}_{E^0A}^{s,t}(Z_2,Z_2)$. (The notation R_j^i here agrees with May's). With this in mind, we may expect that the spectral sequence to be constructed has $E_2\simeq Z_2[\,{}_j^i]$.

Now consider the cobar construction $\bar{c}(A^*)$. We recall that $\bar{c}(A^*)$ is just the tensor algebra $T(I(A^*))$. It is a differential algebra with differential δ given by

$$\delta[\alpha_1|\alpha_2|\dots|\alpha_n]=\sum_{1\leqslant r\leqslant n}\sum_{\mu_r}[\alpha_1|\dots|\alpha_{r-1}|\alpha'_{r,\mu_r}|\alpha''_{r,\mu_r}|\dots|\alpha_n]$$

where $\quad\Delta(\alpha_r)=\sum_{\mu_r}\alpha'_{r,\mu_r}\otimes\alpha''_{r,\mu_r}.$

The cohomology of this differential algebra is $H^*(A)$.

We define a filtration F_pT on $T(I(A^*))$ as follows:

$F_pT^n(I(A^*))=T^n(I(A^*))$ for $p\geqslant -2n$,

$$F_pT^n(I(A^*))=\sum_{\substack{i_1+i_2+\dots+i_n=n+p\\ i_\nu\leqslant -1}}F_{i_1}(I(A^*))\otimes\dots\otimes F_{i_n}(I(A^*))\text{ for }p<-2n.$$

This filtration satisfies the following:

(1) $\dots\subset F_p\subset F_{p+1}\subset\dots$, i.e. the filtration is increasing.

(2) Each F_p is a differential submodule of $T(I(A^*))$. In fact we have $\delta(F_p)\subset F_{p-1}$.

(3) $F_p\cdot F_q\subset F_{p+q}$, i.e. the filtration is multiplicative.

(4) The filtration is complete, i.e.

$\varinjlim_p F_pT=T(I(A^*))$ and $\varprojlim_p T(I(A^*))/F_pT=T(I(A^*))$.

These four properties can be easily checked. For example, the last one follows from the local finiteness of A*.
We establish the resulting spectral sequence by

$$E_0^{p,q}=(F_{-p}/F_{-p-1})_{p+q} \quad \cdots\cdots(*)$$

The reason we define $E_0^{p,q}$ in this way is that $T(I(A^*))$ is a cochain complex, i. e. the differential δ raises the homological degree s by 1 while our filtration is increasing. So if $E_0^{p,q}$ is defined by (*), the spectral sequence $\{E_r\}_{r\geqslant 0}$ will then satisfy the usual rule:

$$\delta_r: E_r^{p,q}\to E_r^{p+r,q-r+1}.$$

From property (3) of the filtration we see the spectral sequence is multiplicative. From property (2) we have $\delta(F_p)\subset F_{p-1}$. So

$$E_1^{p,q}=E_0^{p,q}=(F_{-p}/F_{-p-1})_{p+q}.$$

Now we are going to identify the E_1-term of this spectral sequence with the cobar construction $\bar{c}(E^0A^*)$ as May did in his dissertation.

First we bigrade $\bar{c}(E^0A^*)$ by

$$\bar{c}^{p,q}=\sum_{\substack{i_1+i_2+\cdots+i_n=q\\ p+q=n\\ i_j\leqslant -1}} E^0_{i_1}\otimes\cdots\otimes E^0_{i_n} \quad \cdots\cdots(**)$$

where $\quad E^0_l=E^0_l(A^*)=\bigcup_m E^0_{l,m}(A^*).$

We noted earlier that $\bar{c}(E^0A^*)$ is a differential algebra with differential δ. Also, the E_1-term of our spectral sequence is a differential algebra with differential δ_1 since the spectral sequence is multiplicative. We now compare these two differential algebras.

Proposition 2.1. $\bar{c}(E^0A^*)$, with bigrading defined by (**), is isomorphic to the E_1-term of the spectral sequence as *bigraded differential algebra.*

Proof. Let $x=\xi_1^{i_1}\ldots\xi_n^{i_n}$ be a monomial in A*. We define the length of x to be $l(x)=i_1+\ldots+i_n$. Let M_l be the set of all monomials of length l and $m_l=\bigcup_{t\geqslant l} M_t$. Then we see that m_p is a Z_2-basis of $F_{-p}(A^*)$ and that M_p is a Z_2-basis of $F_{-p}(A^*)/F_{-p-1}(A^*)=E^0_{-p}(A^*)$

$(=\bigcup_m E^0_{-p,m}(A^*))$. Let $K_p^{(n)}$ be the subset of $T^n(I(A^*))$, defined as follows: If $p\leqslant 2n-1$ then $K_p^{(n)}=\phi$. If $p\geqslant 2n$ then $K_p^{(n)}$ is defined to be the set of those $[x_1|x_2|\ldots|x_n]$ such that each x_i is a monomial and such that if $i_\nu=l(x_\nu)$ then $i_1+i_2+\ldots+i_n=p-n$.

Let $\kappa_p^{(n)}=\bigcup_{s\geqslant p} K_s^{(n)}$. Then we see that for each fixed n, $\kappa_p^{(n)}$ is a Z_2-basis of $F_{-p}(T^n)$ and that $K_p^{(n)}$ is a Z_2-basis for $E_1^{p,q}=E_1^{p,q}=F_{-p}(T^n)/F_{-p-1}(T^n)=(F_{-p}T/F_{-p-1}T)_n$ $(p+q=n)$. From the definition of the bigrading of $\bar{c}(E^0A^*)$ it is also clear that $K_p^{(n)}$ is a Z_2-basis of $\bar{c}^{p,q}(E^0A^*)$ $(p+q=n)$. It follows that $\bar{c}(E^0A^*)$ and the E_1-term are naturally isomorphic as bigraded Z_2-modules. Since the algebra structure of $\bar{c}(E^0A^*)$ comes from justposition composition and that of E_1 is induced by the justposition composition of $\bar{c}(A^*)$, it follows that $\bar{c}(E^0A^*)$ and E_1 are also naturally isomorphic as bigraded algebras. It remains to check that their differentials δ and δ_1 agree on their algebra generators. The algebra generators of E_1 are $\{[x]\}$, $x\in A^*$ and those of $\bar{c}(E^0A^*)$ are again $\{[x]\}$ when the x'^s are considered as elements in E^0A^*. It suffices to evaluate $\delta_1([x])$ and $\delta([x])$ for monomials x to see that they are of the same form.

Let $x=\xi_1^{i_1}\ldots\xi_n^{i_n}$, then $[x]\in E_1^{p,q}$ where $i_1+i_2+\ldots+i_n=-q$ and $p+q=1$. It follows that in $\bar{c}(A^*)$

$$\delta([x])=\sum_{\nu_1=0}^{i_1}\ldots\sum_{\nu_n=0}^{i_n}\binom{i_1}{\nu_1}\ldots\binom{i_n}{\nu_n}$$
$$\cdot[\xi_2^{\nu_1}\xi_2^{\nu_2}\ldots\xi_n^{\nu_n}|\xi_1^{i_1-\nu_1}\ldots\xi_n^{i_n-\nu_n}]+\sum_j[x_j'|x_j'']$$

where x_j' and x_j'' are monomials such that

$$l(x_j')+l(x_j'')\geqslant l(x)+1=-q+1.$$

Thus $\sum_j[x_j'|x_j'']\in F_{p-2}$.

Hence,

$$\delta_1([x])=\sum_{\nu_1=0}^{i_1}\sum_{\nu_n=0}^{i_n}\binom{i_1}{\nu_1}\ldots\binom{i_n}{\nu_n}[\xi_1^{\nu_1}\ldots\xi_n^{\nu_n}|\xi_1^{i_1-\nu_1}\ldots\xi_n^{i_n-\nu_n}]. \qquad (1)$$

Now in E^0A^* we have $\Delta(\xi_k)=\xi_k\otimes 1+1\otimes\xi_k$ for each k and hence

$\Delta(\xi_k^m)=\sum_{j=0}^{m}\binom{m}{j}\xi_k^j\otimes\xi_k^{m-j}$. Thus in $\bar{c}(E^0A^*)$ we have

$$\delta([x])=\sum_{\nu_1=0}^{i_1}\cdots\sum_{\nu_n=0}^{i_n}\binom{i_1}{\nu_1}\cdots\binom{i_n}{\nu_n}[\xi_1^{\nu_1}\cdots\xi_n^{\nu_n}|\xi_1^{i_1-\nu_1}\cdots\xi_n^{i_n-\nu_n}]. \quad (2)$$

(1) and (2) show that $\delta([x])=\delta_1([x])$. This is true for all monomials $x\in A^*$ (or in E^0A^*). Thus $\bar{c}(E^0A^*)$ and E_1 are naturally isomorphic as bigraded differential algebras. Q. E. D.

From Proposition 2.1 we have $H(\bar{c}(E^0A^*))\simeq E_2$. But $H(\bar{c}(E^0A^*))$ is just $H^*(E^0A)$ which is isomorphic to $Z_2[R_j^i]$. Thus we have arrived at the conclusion: $E_2\simeq Z_2[R_j^i]$.

We can also construct this spectral sequence for any connected locally finite cocommutative Hopf algebra over Z_2 and get a similar conclusion as above. First we review the structure of any such Hopf algebra.

Let A be a connected locally finite cocommutative Hopf algebra over Z_2 and let A^* be the dual Hopf algebra. By *Theorem 7.11 (Borel Theorem)*, *Proposition 7.8.(3)* of Ref. (2) we see that the algebra structure of A^* must be of the form

$$A^*\simeq Z_2[x_1, x_2, \ldots]/x_k^{2^{\mu_k}}=0$$

where $\{x_k\}$ is a Z_2-basis of indecomposable elements of A^*, $1\leqslant\mu_k\leqslant\infty$, and if $\mu_k=\infty$ we mean x_k has infinite height.

Consider the augmentation filtration F_pA^* of A^* and let E^0A^* be the corresponding associated graded algebra. Then by *Theorem 7.13* of Ref. (2) we have

$$E^0A^*\simeq A^*\simeq Z_2[x_1, x_2, \ldots]/x_k^{2^{\mu_k}}=0 \text{ as algebras.}$$

Again by *Theorem 7.4* of Ref. (2) we have that E^0A^* is primitively generated. So in E^0A^* the coproduct is given by

$$\Delta(x_k)=x_k\otimes 1+1\otimes x_k \text{ all } k.$$

The dual Hopf algebra $E^0A=(E^0A^*)^*$ is, as an algebra, an exterior algebra $E[a_k^i]$ on the duals a_k^i of $x_k^{2^i}$, $i<\mu_k$. The cohomology $H^*(E^0A)$ of E^0A is $Z_2[R_k^i]$ where R_k^i corresponds to a_k^i with $\dim R_k^i=\dim a_k^i$.

Summarizing we have the following

Theorem 2.2. Let A be a connected locally finite cocommutative Hopf algebra over Z_2 and let A* be the dual Hopf algebra. Suppose $A^* \simeq Z_2[x_1, x_2, \ldots]/x_k^{2^{\mu_k}} = 0$ is the algebra structure. Then there exists a multiplicative spectral sequence $\{E_r\}_{r \geq 0}$ which is convergent to the cohomology H*(A) of A and the E_2-term of which is isomorphic to the polynomial algebra $Z_2[R_k^i]$ where R_k^i corresponds to $x_k^{2^i}$ with $i < \mu_k$ and dim R_k^i = dim $x_k^{2^i}$.

§3. APPLICATION TO THE COHOMOLOGY OF RESTRICTED LIE ALGEBRAS

In this section we apply our method to derive May's resolutions of Z_2 over restricted Lie algebras.

Let L be a locally finite restricted Lie algebra over Z_2 with Lie product $[,]: L \otimes L \to L$ and the characteristic map $\xi: L_n \to L_{2n}$ and let V(L) be the universal enveloping algebra of L. V(L) is defined to be the Tensor algebra T(L) modulo the two-sided ideal I where I is the ideal generated by $x \otimes y + y \otimes x + [x. y]$ and $x \otimes x + \xi(x)$. V(L) is a primitively generated Hopf algebra with L as the Z_2-module of primitive elements.

Although it is not stated explicitly in Ref. (2), the following theorem can be easily proved using results given there.

Theorem 3.1. Let A be the connected locally finite cocommutative Hopf algebra over Z_2 and let A* be the dual Hopf algebra. Suppose $A^* \simeq Z_2[x_1, x_2, \ldots]/x_k^{2^{\mu_k}} = 0$ is the algebra structure. Then the following statements are equivalent:

(a) A is primitively generated.

(b) A, as a Hopf algebra, is isomorphic to V(L) for some (locally finite) restricted Lie algebra L.

(c) $\mu_k = 1$ all k, i.e. $A^* \simeq E[x_k]$ as algebras.

Let $\{x_k\}$ be a Z_2-basis of a restricted Lie algebra L. $\{x_k\}$ can be extended to a Z_2-basis m of V(L). Let x_k^* be dual to x_k with respect to m. Then by *Theorem 3.1* we see $V(L)^* \simeq E[x_k^*]$ (as algebras).

From the algebra structure of V(L) we see that the basis m can be chosen so that the coproduct Δ in V(L)* is given by

$$\Delta(x_k^*)=\sum_l x_{\mu_l}^*\otimes x_{\nu_l}^*+\sum_m x_{\lambda_m}^*\otimes x_{\lambda_m}^*+1\otimes x_k^*+x_k^*\otimes 1 \quad \cdots\cdots(*)$$

where l ranges over all (μ_l, ν_l) such that $[x_{\mu_l}, x_{\nu_l}]=x_k$ and such that $\mu_l<\nu_l$ and m ranges over all λ_m such that $x_{\lambda_m}^2=\xi(x_k)$.

Now we apply *Theorem 2.2* to derive May's theorem. If we apply *Theorem 2.2* to the algebra V(L), then since

$$V(L)^*=E[x_k^*]=Z_2[x_1^*, x_2^*, \ldots]/(x_k^*)^2=0$$

it follows that the E_2-term of the spectral sequence becomes $Z_2[R_k^0]$.

Proposition 3.2. (a) $\delta_2: E_2\to E_2$ is given by

$$\delta_2(R_k^0)=\sum_l R_{\mu_l}^0 R_{\nu_l}^0+\sum_m (R_{\lambda_m}^0)^2.$$

(b) $E_3=E_\infty$.

Proof. We prove (b) first.

We have $E_2=Z_2[R_k^0]$. We simply write $R_k=R_k^0$. From the definition of the bigrading of $\bar{c}(E^0A^*)$ we see $R_k\in E_2^{2,-1}$. Hence for each monomial $(R_{k_1})^{\alpha_1}\ldots(R_{k_n})^{\alpha_n}\in Z_2[R_k]$, $(R_{k_1})^{\alpha_1}\ldots(R_{k_n})^{\alpha_n}\in E_2^{p,q}$ where $p=2(\alpha_1+\ldots+\alpha_n)$ and $q=-(\alpha_1+\ldots+\alpha_n)$. Thus if $E_r^{p,q}\neq 0$ then $p=-2q$.

Now suppose $\delta_r: E_r^{p,q}\to E_r^{p+r,q-r+1}$ is a non-zero differential. So $E_r^{p,q}\neq 0$ and $E_r^{p+r,q-r+1}\neq 0$. This implies $p=-2q$ and $p+r=-2(q-r+1)$. From this it follows immediately that $r=2$. Thus the only possible non-zero differential is δ_2. This proves $\delta_r=0$ if $r\geqslant 3$. Thus $E_3=E_\infty$.

Now we prove (a). From the construction of the spectral sequence we see that the E_2-term can be embedded in the cobar construction $\bar{c}(E^0A^*)$. One embedding in this case is given by

$$\phi\left(R_{k_1}^{\alpha_1}\ldots R_{k_n}^{\alpha_n}\right)$$
$$=[x_{k_1}^*|\ldots|x_{k_1}^*|x_{k_2}^*|\ldots|x_{k_2}^*|\ldots|x_{k_n}^*|\ldots|x_{k_n}^*] \qquad (k_1<\ldots<k_n).$$
$$\longleftarrow \alpha_1 \longrightarrow \longleftarrow \alpha_2 \longrightarrow \quad \longleftarrow \alpha_n \longrightarrow$$

So to evaluate $\delta_2(R_k)$ it suffices to evaluate $\delta([x_k^*])$ in $\bar{c}(V(L)^*)$. Now from (*) we have

$$\delta([x_k^*])=\sum_l[x_{\mu_l}^*|x_{\nu_l}^*]+\sum_m[x_{\lambda_m}^*|x_{\lambda_m}^*].$$

Since $[x_k^*]\in F_{-2}/F_{-3}$ and $[x_{\mu_l}^*|x_{\nu_l}^*]$, $[x_{\lambda_m}^*|x_{\lambda_m}^*]\in F_{-4}/F_{-5}$ it follows that $\delta_2(R_k)=\sum_l R_{\mu_l}R_{\nu_l}+\sum_m R_{\lambda_m}^2$. This completes the proof of (a). Q.E.D. From proposition 3.2 we observe that E_∞ is the homology of the differential algebra $E_2=Z_2[R_k]$ with differential δ_2 given by $\delta_2(R_k)=\sum_l R_{\mu_l}R_{\nu_l}+\sum_m R_{\lambda_m}^2$. In this case we can check that the algebra extension from E_∞ to $H^*(V(L))$ is trivial. In this way we have arrived at the following conclusion.

Theorem 3.3. $H^*(V(L))$ is the homology of the differential algebra E_2.

Here E_2 is exactly the complex $\bar{X}^{*(1)}$ derived from May's resolution of Z_2 over $V(L)$.

REFERENCES

(1) J.P. May, Cohomology of restricted Lie algebras and of Hopf algebras. Application to the Steenrod algebra; *Ph. D. Dissertation, Princeton*, 1964.

(2) J. Milnor and J.C. Moore, On the structure of Hopf algebras. *Ann. of Math.*, vol 81, 1965.

"If I have seen farther than others, it is by standing on the shoulders of giants."

An old saying, quoted by *Newton.*

APPLICATIONS OF SPECTRAL SEQUENCES (I)*

(Homology and Cohomology of a Monoid)

HON-YU MA

ABSTRACT

We construct the homology group $H(\pi, X)$ of a group π with coefficients in space on which π operates. Under some modifications, we obtain a covering space $X \to X/\pi$. Then we construct the spectral sequence for this covering space, and finish the correspondence between $H(\pi, X)$ and $H(X_\pi)$. The cohomology groups are considered in the same way as homology groups. Some applications are given, e.g. if a group of two elements acts upon an n-dimensional sphere, one computes the cohomology group of an n-dimensional projective space.

§1. HOMOLOGY GROUPS OF π WITH COEFFICIENTS IN A $Z(\pi)$-MODULE

Let π be a monoid, and $C=\{C_q, d_q\}$ be a chain complex of free left $Z(\pi)$-modules with argumentation $\varepsilon: C_0 \to Z$ (where Z has the trivial $Z(\pi)$-module structure) such that the sequence $\cdots\to C_q \to C_{q-1} \to \cdots \to C_1 \to C_0 \to Z \to 0$ is exact, i.e. C is acyclic. Then for any right $Z(\pi)$-module A, the sequence: $\cdots \to A \underset{\pi}{\otimes} C_q \to \cdots A \underset{\pi}{\otimes} C_1 \to A \underset{\pi}{\otimes} C_0 \to 0$ define a chain complex with homology groups $H_q(A \underset{\pi}{\otimes} C)$.

Furthermore, let f: $A \to B$ be a $Z(\pi)$-homorphism, then it induces homomorphism f_*: $H_q(A \underset{\pi}{\otimes} C) \to H_q(B \underset{\pi}{\otimes} C)$, and for an exact sequence $0 \to A' \to A \to A'' \to 0$ of $Z(\pi)$-modules, we have the exact sequence $0 \to A' \underset{\pi}{\otimes} C \to A \underset{\pi}{\otimes} C \to A'' \underset{\pi}{\otimes} C \to 0$. From this there exists a canonical homorphism ∂: $H_q(A'' \underset{\pi}{\otimes} C) \to H_{q-1}(A' \underset{\pi}{\otimes} C)$. Thus we have a homology theory

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$\{H_q(A \underset{\pi}{\otimes} C), f_*, \partial\}$ for the monoid π. $H_q(A \underset{\pi}{\otimes} C)$ is called the homology group of π with coefficient in A. They are independent on the choice of $C^{(1)}$. Therefore, we denote them simply by $H_q(\pi, A)$.

Theorem 1. (1) $H_0(\pi, A) = A_\pi$ where A_π is the quotient module of A by the submodule generated by all the elements of the form:

$$ax - a \qquad a \in A,\ x \in \pi.$$

(2) $H_q(\pi, A) = 0$ for $q > 0$ if A is a free $Z(\pi)$-module.

Proof. (1) Since the sequence $C_1 \to C_0 \to Z \to 0$ is exact, thus we have: $H_0(A \underset{\pi}{\otimes} C) = A \underset{\pi}{\otimes} C_0 / \mathrm{Im}(A \underset{\pi}{\otimes} C_1 \to A \underset{\pi}{\otimes} C_0)$

$$= A \underset{\pi}{\otimes} C_0 / \ker(A \underset{\pi}{\otimes} C_0 \to A \underset{\pi}{\otimes} Z) = A \underset{\pi}{\otimes} Z.$$

While $\{ax - a: a \in A, x \in \pi\}$ is the kernel of the canonical homomorphism $A \to A \underset{\pi}{\otimes} Z$ by $a \to a \otimes 1$,

hence then $A \underset{\pi}{\otimes} Z = A/\{ax - a: a \in A, x \in \pi\} = A_\pi$.

(2) Since A is a free $Z(\pi)$-module, we consider $Z(\pi)$ as a right $Z(\pi)$-module. Then A can be considered as the direct sum of certain numbers of copies of $Z(\pi)$. Then the canonical isomorphisms: $Z(\pi) \underset{\pi}{\otimes} C_q \cong Z \otimes C_q \cong C_q$ implying $H_q(Z(\pi) \underset{\pi}{\otimes} C) = 0$ for $q > 0$. Therefore, we have $H_q(A \underset{\pi}{\otimes} C) = 0$ for $q > 0$. Q. E. D.

§2. COHOMOLOGY GROUPS OF π WITH COEFFICIENTS IN A $Z(\pi)$-MODULE

We consider a left $Z(\pi)$-module A, and let $\mathrm{Hom}_\pi(C_q, A)$ be the group of the $Z(\pi)$-homomorphisms from C_q to A. Composing every element of $\mathrm{Hom}_\pi(C_q, A)$ with $d: C_{q+1} \to C_q$ gives an element of $\mathrm{Hom}_\pi(C_{q+1}, A)$. Then there exists a homomorphism $d': \mathrm{Hom}_\pi(C_q, A) \to \mathrm{Hom}_\pi(C_{q+1}, A)$ with $d'd' = 0$. Therefore we obtain a cochain complex $\mathrm{Hom}_\pi(C, A)$ with cohomology groups $H^q(\mathrm{Hom}_\pi(C, A))$. For a given homomorphism $f: A \to B$, there exists $f^*: H^q(\mathrm{Hom}_\pi(C, A)) \to H^q(\mathrm{Hom}_\pi(C, B))$, and for a given exact sequence $0 \to A' \to A \to A'' \to 0$, there is an exact sequence: $0 \to \mathrm{Hom}_\pi(C, A') \to \mathrm{Hom}_\pi(C, A) \to \mathrm{Hom}_\pi(C, A'') \to 0$, and there exists a canonical homomorphism $\delta: H^q(\mathrm{Hom}_\pi(C, A'')) \to H^{q+1}(\mathrm{Hom}_\pi(C, A'))$. Therefore we have a cohomology theory for the

monoid π, say $H^*=\{H^q(\mathrm{Hom}_\pi(C, A)), f^*, \delta\}$ and denote $H^q(\mathrm{Hom}_\pi(C, A))$ simply by $H^q(\pi, A)$ also.

Theorem 1′. (1) $H^0(\pi, A)=A^\pi$ where $A^\pi=\{a\in A: xa=a$ for any $x\in\pi\}$

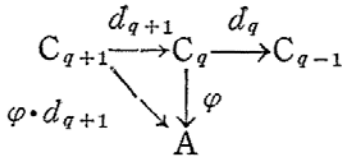
(2) $H^q(\pi, A)=0$ for $q>0$, if A is π-injective.

Proof. (1) Since the sequence $C_1\to C_0\to Z\to 0$ is exact, thus $\mathrm{Hom}_\pi(C_1, A)\leftarrow\mathrm{Hom}_\pi(C_0, A)\leftarrow\mathrm{Hom}_\pi(Z, A)\leftarrow 0$ is exact. Hence then

$$\begin{aligned}H^0(\mathrm{Hom}_\pi(C, A))&=\ker(\mathrm{Hom}_\pi(C_0, A)\to\mathrm{Hom}_\pi(C_1, A))\\&=\mathrm{Im}(\mathrm{Hom}_\pi(Z, A)\to\mathrm{Hom}_\pi(C_0, A))\\&\cong\mathrm{Hom}_\pi(Z, A).\end{aligned}$$

And we can identify any homomorphism $\varphi: Z\to A$ with $\varphi(1)$, and then get the result $\mathrm{Hom}_\pi(Z, A)=A^\pi$.

(2) Let $\varphi: C_q\to A$ $(q>0)$ such that $\varphi\cdot d_{q+1}=0$, we have that:



where $\ker d_q=\mathrm{Im}\ d_{q+1}\subset\ker\varphi$. Passing to the quotient, there exists $\varphi^*: C_q/\mathrm{Im}\ d_{q+1}\to A$. Hence φ defines a homomorphism ψ from the image of d_q to A, and A becomes π-injective. Then ψ can be extended to $C_{q-1}\to A$, which also be denoted by ψ. We have $\psi\cdot d_q=\varphi$. Therefore $H^q(\mathrm{Hom}_\pi(C, A))=0$ for $q>0$. Q. E. D.

§3. SPACE WITH GROUP OPERATION

Let $\boldsymbol{X}$ be topological space, and $\mathrm{Aut}(\boldsymbol{X})$ be the group of homeomorphisms from $\boldsymbol{X}$ into itself. We say that a group π operates on $\boldsymbol{X}$ if there exists a group homomorphism $\mu: \pi\to\mathrm{Aut}(\boldsymbol{X})$. An open set U in $\boldsymbol{X}$ is said to be proper (under the action of π) if $a\mathrm{U}\cap\mathrm{U}=\phi$ whenever $a\in\pi$ and $a\neq 1$. A group π is said to operate properly on $\boldsymbol{X}$ if every point of $\boldsymbol{X}$ is contained in a proper open set which constitutes a base for the topology of $\boldsymbol{X}$. When π operates properly on $\boldsymbol{X}$, then no homeomorphism $\mu(a)$ with $a\neq 1$ can leave any point fixed; we say that π operates on $\boldsymbol{X}$ without fixed point. Let p be

the projection of X to its orbit space X/π. Then X is a covering space of X/π.

§4. HOMOLOGY GROUP OF π WITH COEFFICIENTS IN A GRADED COMPLEX WHERE π OPERATES ON THE RIGHT

Let X be the singular chain complex of X on which π operates from the right, and let C be a space on which π operates from the left, and suppose that the homology of C is trivial (i.e. $H_q(C)=0$ for $q>0$, $H_0(C)\cong Z$), and the chain complex C of C is formed by free $Z(\pi)$-modules. Consider the tensor product $X \underset{\pi}{\otimes} C$ as the quotient of $X \underset{Z}{\otimes} C$ by the equivalence relation defined by $(x\cdot s^{-1}) \otimes s\cdot y \equiv x \otimes y$ for any $x \in X$, $y \in C$ and $s \in \pi$. The graded group $X \underset{\pi}{\otimes} C$ is bigraded by graduations of X and of C. An element $x \otimes y$ has total degree $p+q$ if x is of degree p and y is of degree q. We define a new boundary operator on $X \underset{\pi}{\otimes} C$, i.e.

$\partial(x \otimes y)=(\partial_X x) \otimes y+(-1)^p x \otimes \partial_C y$ if p is the degree of x in X. Then $X \underset{\pi}{\otimes} C$ is a graded complex. Thus we can obtain the homology groups $H_q(X \underset{\pi}{\otimes} C)$ which are independent of the choice of C and we denote them simply by $H_q(\pi, X)$.

§5. COHOMOLOGY GROUPS OF π WITH COEFFICIENTS IN A GRADED DIFFERENTIAL ALGEBRA WHERE π OPERATES ON THE LEFT

Let Y be the singular cochain complex of the space X, let π operate on the left of Y, and C be the same as in the homology case. Consider the group $\mathrm{Hom}_\pi(C, Y)$ which is composed of bigraded groups $\mathrm{Hom}_\pi(C_p, Y^q)$.

We define a differential operator δ on $\mathrm{Hom}_\pi(C, Y)$ by

$\delta f=\partial_c^* f+(-1)^p \delta_Y * f$ for any $f \in \mathrm{Hom}_\pi(C^p, Y^q)$

where ∂_c^*: $\mathrm{Hom}_\pi(C_p, Y^q) \to \mathrm{Hom}_\pi(C_{p+1}, Y^q)$ is induced by ∂_c and $\partial_c^* f=f\circ\partial_c$

$\delta_Y *$: $\mathrm{Hom}_\pi(C_p, Y^q) \to \mathrm{Hom}_\pi(C_p, Y^{q+1})$ is induced by δ_Y and $\delta_Y * f=\delta_Y \circ f$

Then it is clear that $\delta\delta=0$; thus we can take cohomology; say $H^*(\mathrm{Hom}_\pi(C, Y))$, which is also independent of the choice of $\boldsymbol{C}$. We also denote it simply by $H^*(\pi, Y)$.

§6. THE SUFFICIENT CONDITION FOR $H(\pi, X)\to H(X_\pi)$ TO BE ISOMORPHIC

I. First case: The elements of X are all of degree 0.

To say that β^*: $H(\pi, X)\to H(X_\pi)$ induced from $X\underset{\pi}{\otimes}C\xrightarrow{\beta}X_\pi=X\underset{\pi}{\otimes}Z$, where β is the homomorphism $C_1\to Z$ by composing $C_1\to C_0$ and $C_0\to Z$, is an isomorphism onto, is equivalent to that $H_q(\pi, X)$ is null for $q\geq 1$ and $H_0(\pi, X)=X_\pi$ (the latter has been proven above).

If X is π-free, then we have already that β^* is an onto isomorphism. Next, we consider the weaker condition as following:

Define a mapping φ: $X\underset{Z}{\otimes}Z(\pi)\to X$ by $x\otimes\lambda\to x\lambda$. If we consider $X\underset{Z}{\otimes}Z(\pi)$ as a right $Z(\pi)$-module defined by the second factor $Z(\pi)$, then φ becomes a $Z(\pi)$-homomorphism. We say that X is weakly π-free if $\ker\varphi$ is a direct summand; in other words, there exists a $Z(\pi)$-homomorphism $X\to X\underset{Z}{\otimes}Z(\pi)$, which if composed with φ gives the identity automorphism of X.

Proposition 1. If X is weakly π-free, then the homology group $H_q(\pi, X)$ is null for $q\geq 1$, i.e. β^* is an onto isomorphism.

Proof. From the identity map $X\to X\underset{Z}{\otimes}Z(\pi)\to X$, we deduce the identity map $X\underset{\pi}{\otimes}C\to(X\underset{Z}{\otimes}Z(\pi))\underset{\pi}{\otimes}C\to X\underset{\pi}{\otimes}C$, i.e.

$X\underset{\pi}{\otimes}C\to X\underset{Z}{\otimes}C\to X\underset{\pi}{\otimes}C$ is the identity map[3]. Passing to the homology, we have that:

$H_q(\pi, X)\to H_q(X\underset{Z}{\otimes}C)\to H_q(\pi, X)$ is the identity homomorphism. But $H_q(X\underset{Z}{\otimes}C)$ is null for $q\geq 1$, since $H_q(C)=0$ for $q\geq 1$[1]. Therefore $H_q(\pi, X)=0$ for $q\geq 1$. Q.E.D.

II. General case: Not all elements of X are of degree 0.

Let X' be the subjacent graded group of the complex X. By "X is weakly π-free", we mean that for each integer n, X'_n is weakly π-free. That is we have: $H_q(\pi, X'_n)=0$ for $q\geq 1$ and any n. (*)

Theorem 2. Under the condition (*), β_*: $H(\pi, X)\to H(X_\pi)$ is an onto isomorphism.

Proof. Let $A=X\underset{\pi}{\otimes}C$, $B=X\underset{\pi}{\otimes}Z$, $A_p=\sum_{n\leqslant p}X_n\underset{\pi}{\otimes}C$ and $B_p=\sum_{n\leqslant p}X_n\underset{\pi}{\otimes}Z$. Then β maps A_p into B_p, and induces a homomorphism: $A_p/A_{p-1}\to B_p/B_{p-1}$ which is compatible with the boundary operator, while $A_p/A_{p-1}=X_p\underset{\pi}{\otimes}C$, $B_p/B_{p-1}=X_p\underset{\pi}{\otimes}Z$. Thus the homomorphism $H(A_p/A_{p-1})\to H(B_p/B_{p-1})$ is an onto isomorphism. Then by the following two lemmas, β is an onto isomorphism.

Lemma 1. Let A and B be two abelian groups with "boundary operators" and f is a homomorphism from A to B which is compatible with the boundary operators, and $\cdots\subset A_p\subset A_{p+1}\subset\cdots$, $\cdots\subset B_p\subset B_{p+1}\subset\cdots$ be two filtrations of A and B respectively. Suppose also that $f(A_p)\subset B_p$ for any integer p and $H(A_{p+1}/A_p)\to H(B_{p+1}/B_p)$ defined by f is an onto isomorphism. Then for any integer $q\geq 1$ and any integer p

$H(A_{p+q}/A_p)\to H(B_{p+q}/B_p)$ is an onto isomorphism.

Proof. By induction on q, it is true for $q=1$. Suppose that it is also true for q, we prove that it is true for $q+1$.

The exact sequences: $0\to A_{p+q}/A_p\to A_{p+q+1}/A_p\to A_{p+q+1}/A_{p+q}\to 0$

and $0\to B_{p+q}/B_p\to B_{p+q+1}/B_p\to B_{p+q+1}/B_{p+q}\to 0$

define two exact sequences for the homology groups and induce a homomorphism from the former to the latter, i.e. we have the following commutative diagram:

$$\begin{array}{ccccccccc}
H(A_{p+q+1}/A_{p+q}) & \to & H(A_{p+q}/A_p) & \to & H(A_{p+q+1}/A_p) & \to & H(A_{p+q+1}/A_{p+q}) & \to & \\
\downarrow\wr & & \downarrow\wr & & & & \downarrow\wr & & \\
H(B_{p+q+1}/B_{p+q}) & \to & H(B_{p+q}/B_p) & \to & H(B_{p+q+1}/B_p) & \to & H(B_{p+q+1}/B_{p+q}) & \to & \\
H(A_{p+q}/A_p) & & & & & & & & \\
\downarrow\wr & & & & & & & & \\
H(B_{p+q}/B_p). & & & & & & & &
\end{array}$$

By five lemma[1], we have that $H(A_{p+q+1}/A_p)\cong H(B_{p+q+1}/B_p)$.

Q. E. D.

Lemma 2. Under the hypothesis of Lemma 1, suppose also that A and B are graded groups, i.e. $A=\sum_n A^n$, $B=\sum_n B^n$, and the graduations are compatible with the filtrations (for example: $A^n_p=A^n\cap A_p$ for the filtration A^n_p of A^n), and for any n, $A^n_p=0$ if p is sufficiently small

$$A_p^n = A^n \text{ if } p \text{ is sufficiently large.}$$

Then the homomorphism $H(A)\to H(B)$ defined by f is an onto isomorphism.

Proof. In fact, it suffices to verify that $H(A^n)\to H(B^n)$ is an onto isomorphism for any n. But this result follows from Lemma 1 immediately.

§7. THE SUFFICIENT CONDITION FOR $H^*(Y^\pi)\to H^*(\pi, Y)$ TO BE AN ISOMORPHISM

I. First case: The elements of the left π-module Y are all of degree 0.

Under this condition, the homomorphism $H^*(Y^\pi)\to H^*(\pi, Y)$ induced by $\mathrm{Hom}_\pi(Z, Y)\to\mathrm{Hom}_\pi(C, Y)$ is an onto isomorphism, that is $H^q(\pi, Y)=0$ for any $q\geq 1$ and $H^*(Y^\pi)=Y^\pi=H^0(\pi, Y)$ (the latter has been proven in Th. 1').

We consider the map ψ: $Y\to\mathrm{Hom}_z(Z(\pi), Y)$ by

$$y\to hy \qquad \text{where}$$

$$hy\colon\ Z(\pi)\to Y \text{ by}$$

$$\lambda\to\lambda y,$$

then ψ is a 1-1 $Z(\pi)$-homomorphism, if $\mathrm{Hom}_z(Z(\pi), Y)$ is considered as a left $Z(\pi)$-module defined by the structure of the right $Z(\pi)$-module $Z(\pi)$. Furthermore, we say that Y is weakly π-injective if there exists a $Z(\pi)$-homomorphism $\mathrm{Hom}_z(Z(\pi), Y)\to Y$, and composing it with ψ gives the identity automorphism of Y.

Proposition 1'. If Y is weakly π-injective, then $H^q(\pi, Y)=0$ for $q\geq 1$.

II. Second case: Not all elements of Y are of degree 0.

Theorem 2'. Let Y be a graded complex with coboundary operator of degree +1, Y' be its subjacent graded group. Suppose also that

$$H^q(\pi, Y'_n)=0 \text{ for } q\geq 1 \text{ and for any } n.$$

Then $H^*(Y^\pi)\to H^*(\pi, Y)$ is an onto isomorphism.

Proof. It is similar to the proofs in the homology case.

§8. THE SPECTRAL SEQUENCES

We now return to the discussion of §4. Given a filtration structure

on $X \underset{\pi}{\otimes} C$: $L_p = \sum_{n \leqslant p} X \underset{\pi}{\otimes} C_n$, $L_p \subset L_{p+1}$, there is a spectral sequence[3] such that: $E^1 = \sum_p H(L_p/L_{p-1})$, $E^2 = H(E^1), \cdots, E^{k+1} = H(E^k), \cdots, E^\infty$. The homologies of $E^1, E^2, \cdots$ are relative to the operators $d^1, d^2, \cdots$ respectively, and the term E^∞ is associated with the graded $H(\pi, X)$.

Let $E^k_{n,p}$ denote the subgroup of E^k formed by the terms of total degree n, and the filtration degree p. Then we have that:

$$E^k_{n,p} = 0 \text{ for } p<0 \text{ or } p>n \text{ and}$$

$$E^\infty_{n,p} = E^k_{n,p} \text{ for } k > \max\{p, n+1-p\}.$$

Theorem 3. $E^2_{n,p} \cong H_p(\pi, H_{n-p}(X))$, where π operates on the right of $H_{n-p}(X)$.

Proof. Since $E^1 = \sum_p H(L_p/L_{p-1})$ and $H(L_p/L_{p-1})$ is the homology group of $X \underset{\pi}{\otimes} C_p$, where the boundary operator is defined by the operator ∂_X of X, and C_p is a free $Z(\pi)$-module, thus $H(X \underset{\pi}{\otimes} C_p) = H(X) \underset{\pi}{\otimes} C_p$[1]. Therefore $E^1_{n,p} \cong H_{n-p}(X) \underset{\pi}{\otimes} C_p$. Since the boundary d^1 of $E^1 = H(X) \underset{\pi}{\otimes} C$ is defined by the boundary of C:

$$\partial(\gamma \otimes c) = (-1)^{n-p} \gamma \otimes \partial_c \quad \text{for any } \gamma \in H_{n-p}(X).$$

Therefore $E^2 = H(E^1)$ is the homology group of π with coefficient in $H(X)$ where π operates on the right.

Now we refer to the case of §5.

Let $L'_p = \sum_{n \geqslant p} \mathrm{Hom}_\pi(C_n, Y)$, then $\{L'_p\}$ forms a decreasing filtration for $\mathrm{Hom}_\pi(C, Y)$ and there is a spectral sequence:
$E^1 = \sum_p H(L'_p/L'_{p+1})$, $E^2 = H(E^1), \cdots, E^{k+1} = H(E^k), \cdots, E^\infty$, and E^∞ is associated with the graded group $H^*(\pi, Y)$.

Theorem 3′. $E^2_{n,p} \cong H^p(\pi, H^{n-p}(Y))$, where π operates on the left of $H^{n-p}(Y)$.

Proof. The same as Theorem 3.

§9. THE APPLICATIONS OF THE SPECTRAL SEQUENCES

I. Application to the case: For any n, there exists an integer p_n such that $E^2_{n,p} = 0$ for $p \neq p_n$, and $p_{n+1} - p_n \leq 1$.

Theorem 4. Under the above hypothesis, $H_n(\pi, X)$ is isomorphic to E^2_{n,p_n}.

Proof. By the hypothesis, all of $d^2, d^3, \cdots$ are null. Thus the subgroup of E^∞ formed by the elements with total degree n identifies to E^2_{n,p_n}. We have that $E^\infty_{n,p_n} = B_{n,p_n}/B_{n,p_n-1}$, where $B_{n,p} = \mathrm{Im}(H_n(L_p) \to H_n(\pi, X))$. $0 \subseteq \underset{\underset{0}{\|}}{B_{n,0}} \subseteq \cdots \subseteq \underset{\underset{0}{\|}}{B_{n,p_n-1}} \subseteq B_{n,p_n} = H_n(\pi, X)$.

Hence then $H_n(\pi, X) = E^2_{n,p_n}$. Q. E. D.

Theorem 5. A group π operating in a finite dimensional acyclic manifold without fixed point, does not contain an element ($\neq 1$) of finite order.

Proof. Let $\boldsymbol{X}$ be this manifold, and X be its chain complex, then $H_q(X) = 0$ for $q \geq 1$. Therefore by Theorem 4, we have that $H_n(\pi, X) \cong H_n(\pi, H_0(X))$. Since the group π operates on $\boldsymbol{X}$ without fixed point, thus by Theorem 2, we have that $H_q(\pi, X) \cong H_q(X_\pi)$. But $\boldsymbol{X}/\pi$ is of dimension n, hence $H_q(X_\pi) = 0$ for $q > n$. Then $H_q(\pi, Z) \cong H_q(\pi, H_0(X)) \cong H_q(\pi, X) \cong H_q(X_\pi)$, i. e. $H_q(\pi, Z) \cong H_q(X_\pi)$.

This shows that π can not be a nontrivial cyclic group of finite order, since for such group, there exist infinite many values of q such that $H_q(\pi, Z) \neq 0$. For example: $H_{2p+1}(\pi, Z) \cong Z/hZ$, where h is the order of π[4]. Q. E. D.

II. Application to the case: $H_q(X) = 0$ for $0 < q < n$.

For this case, we have that: $E^2_{q,p} \cong H_p(\pi, H_{q-p}(X)) = 0$ for all $0 \leq q < n$ with $q \neq p$, and $H_q(\pi, H_0(X)) \cong E^2_{q,q} \cong E^\infty_{q,q}$ for $q < n$, and $0 = B_{q,0} = \cdots = B_{q,q-1} \subseteq B_{q,q} \cong H_q(\pi, X)$.

Thus we have that $H_q(\pi, X) \cong H_q(\pi, H_0(X))$ for $q < n$.

We have also the following two diagrams:

$$\left.\begin{array}{rl}
0 = \left\{\begin{array}{l} E^2_{n+1,2} \longrightarrow E^2_{n,0} \longrightarrow E^2_{n-1,-2} \\ \qquad\qquad\quad \| \\ \qquad\qquad\quad \vdots \\ \qquad\qquad\quad \| \end{array}\right. & \\
H_{n+1}(\pi, H_0(X)) = E^{n+1}_{n+1,n+1} \longrightarrow E^{n+1}_{n,0} & \\
\qquad\qquad\qquad\qquad\qquad \downarrow p_r & \\
0 = \left\{\begin{array}{l} E^{n+2}_{n+1,n+2} \longrightarrow E^{n+2}_{n,0} \\ \qquad\qquad\quad \| \\ \qquad\qquad\quad \vdots \\ \qquad\qquad\quad \| \\ \qquad\qquad\quad E^\infty_{n,0} \end{array}\right. &
\end{array}\right\} = 0$$

and

$$0=\begin{cases} \left.\begin{array}{ccc} E^2_{n+2,n+3} & \longrightarrow E^2_{n+1,n+1} & \longrightarrow E^2_{n,n-1} \\ & \| & \\ & \vdots & \\ & \| & \end{array}\right\}=0 \\ \qquad\qquad E^{n+1}_{n+1,n+1} \xrightarrow{d^{n+1}} E^{n+1}_{n,0}=H_0(\pi, H_n(X)) \\ \qquad\qquad \uparrow i \\ \left.\begin{array}{ccc} E^{n+2}_{n+1,n+1} & \longrightarrow E^{n+2}_{n,-1} \\ \| & \\ \vdots & \\ \| & \\ E^{\infty}_{n+1,n+1} & \end{array}\right\}=0 \end{cases}$$

From the above diagrams, we can derive the following exact sequence:

$$E_{n+1,n+1}\cong E^{n+2}_{n+1,n+1}\xrightarrow{i}E^{n+1}_{n+1,n+1}\xrightarrow{d^{n+1}}E^{n+1}_{n,0}\xrightarrow{p_r}E_{n,0}\to 0.$$

Therefore:

$$H_{n+1}(\pi, X)\xrightarrow{p'_r}H_{n+1}(\pi, X)/B_{n+1,1}\xrightarrow{i}H_{n+1}(\pi, H_0(X))\xrightarrow{d^{n+1}}H_0(\pi, H(X))$$
$$\xrightarrow{p_r}E_{n,0}\xrightarrow{i'}H_n(\pi, X)\xrightarrow{p''_r}H_n(\pi,X)/E_{n,0}\to 0 \text{ is exact.}$$

i. e.

$$H_{n+1}(\pi, X)\xrightarrow{i\cdot p'_r}H_{n+1}(\pi, H_0(X))\xrightarrow{d^{n+1}}(H_n(X))_\pi\xrightarrow{i'\cdot p_r}H_n(\pi, X)$$
$$\xrightarrow{p''_r}H_n(\pi, H_0(X))\to 0 \text{ is exact.}$$

III. Application to the case: $H^q(Y)=0$ for $0<q<n$.

Under this condition, we have $H^q(\pi, Y)\cong H^q(\pi,H^0(Y))$ for $0\leqslant q<n$, and then an exact sequence:

$$0\to H^n(\pi, H^0(Y))\to H^n(\pi, Y)\to (H^n(Y))^\pi\to H^{n+1}(\pi, H^0(Y))\to H^{n+1}(\pi, Y)$$

§10. APPLICATION OF §9 TO THE SPHERE S^n

Let π be a group operating properly on S^n, and all ordinal cohomology groups be with integers as coefficients. From §9, III, we have:

$$H^q(\pi, S^n)\cong H^q(S^n/\pi)\cong H^q(\pi, Z) \text{ for } q<n \qquad (1)$$

and the exact sequence:

$$0\to H^n(\pi,Z)\to H^n(S^n/\pi)\to(H^n(S^n))^\pi\to H^{n+1}(\pi,Z)\to 0 \qquad (2)$$

Examining for the case that π is the group with two elements formed by the identity homeomorphism and the antipodal homeomorphism, by connecting the center, we have that $S^n/\pi\cong P^n$.

Next, we want to compute the cohomology groups of P^n with integers as coefficients. From (1), we have directly that:

$$H^q(P^n)\cong H^q(\pi,Z) \qquad \text{for } q<n.$$

While $H^q(\pi,Z)\cong Z_2$ if q is even

$H^q(\pi,Z)=0$ if q is odd[4].

Hence $$H^q(P^n)\begin{cases}\cong Z_2 & \text{if } q \text{ is even and } q<n,\\ =0 & \text{if } q \text{ is odd and } q<n.\end{cases}$$

To compute $H^n(P^n)$:

(i) If n is odd, then the antipodal homeomorphism of S^n preserves the orientation of S^n. Thus any element of $H^n(S^n)$ is invariant under π, and then (2) becomes the exact sequence:

$$0\to H^n(P^n)\to H^n(S^n)\to Z_2\to 0$$
$$\qquad\qquad\quad \| \wr$$
$$\qquad\qquad\quad Z$$

Hence $H^n(P^n)\cong Z$, and the map $H^n(P^n)\to H^n(S^n)$ is given by $m\to 2m$.

(ii) If n is even, then the antipodal homeomorphism reverses the orientation of S^n. Therefore $(H^n(S^n))^\pi=0$, and the sequence

$0\to Z_2\to H_n(P^n)\to 0$ is exact,

hence $H^n(P^n)=Z_2$.

REFERENCES

(1) E. H., Spanier, *Algebraic Topology*, McGraw-Hill (1966).
(2) S., MacLane, *Homology*, Academic Press (1963).
(3) J. P., Serre, Homologie Singuliere des Espace Fibres, *Ann. of Math.* **54**: (1951).
(4) H. Cartan, and S. Eilenberg, *Homological Algebra*, Princeton Press (1956).

"If little knowledge is dangerous, where is the man who has so much as to be out of danger?"

T. H. Huxley, Science and Culture: On Elementary Instruction in Physiology (1881).

APPLICATIONS OF SPECTRAL SEQUENCES (II)*

(Sheaf Theory)

HON-YU MA

ABSTRACT

In this paper we take up the topic: Sheaf Theory. The terms pre-sheaf and sheaf are first defined, and some special sheaves are described along with a discussion of properties that will be useful in later applications. For every Theorem or Proposition a complete proof is given, together with constructions of Sheaf Homology and Cohomology. Then the spectral sequence of a differential sheaf is applied to the Leray Sheaf. Finally, the method is applied to locally trivial bundles and is used to get Wang and Gysin exact sequences.

§1. PRELIMARIES CONCERNING PRESHEAVES AND SHEAVES

A presheaf A on a topological space X is a function which assigns to each open set U of X an abelian group A(U); and to each pair U, V of open sets with $U \subset V$, a homomorphism (restriction) $r_{U,V}$: $A(V) \to A(U)$ in such a way that:

$$r_{U,U}=1, \text{ and } r_{U,V}r_{V,W}=r_{U,W} \text{ when } U \subset V \subset W.$$

A homomorphism h: $A \to B$ of presheaves is a collection of homomorphisms h_U: $A(U) \to B(U)$ commuting with restrictions.

A sheaf of abelian groups on the space X is a pair $(\mathbf{A}, \pi)$, where

i) $\mathbf{A}$ is a topological space,

ii) π: $\mathbf{A} \to X$ is a local homeomorphism onto X,

iii) For each x in X, $\mathbf{A}_x=\pi^{-1}(x)$ is an abelian group and is called the stalk of $\mathbf{A}$ at x,

iv) The group operations are continuous. That means the map $\alpha \mapsto -\alpha$ of $\mathbf{A} \to \mathbf{A}$ and the map $(\alpha, \beta) \mapsto \alpha+\beta$ of the subspace

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$\mathbf{A}\Delta\mathbf{A}$ consisting of those pairs (α, β) with $\pi(\alpha)=\pi(\beta)$ to $\mathbf{A}$ are continuous.

And sheaves of rings can be defined in the same way with some modifications.

A homomorphism h: $\mathbf{A}\to\mathbf{B}$ of sheaves is a continuous map with $h(\mathbf{A}_x)\subset\mathbf{B}_x$ for each x in X, and the restriction h_x: $\mathbf{A}_x\to\mathbf{B}_x$ of h to stalks is a homomorphism for all x. If $0\to\mathbf{A}'\to\mathbf{A}\to\mathbf{A}''$ is an exact sequence of sheaves, then the induced sequence $0\to\mathbf{A}'(\mathrm{Y})\to\mathbf{A}(\mathrm{Y})\to\mathbf{A}''(\mathrm{Y})$ of sections over any subset Y of X is exact.

Given a presheaf A, let $\mathbf{A}_x=\varinjlim \mathrm{A}(\mathrm{U})$, where U ranges over the open neighborhoods of x in X, and let $\mathbf{A}$ be the union of groups $\mathbf{A}_x$ for all x, π: $\mathbf{A}\to\mathrm{X}$ be the map mapping each point of $\mathbf{A}_x$ to x. For a fixed element s in A(U), we take the set of all germs $s_x\in\mathbf{A}_x$ for $x\in\mathrm{U}$ as an open set in $\mathbf{A}$. Then the set $\mathbf{A}$ with the topology generated by these open sets is called the sheaf generated by the presheaf A.

1.1 **Proposition.** If $\mathbf{A}_0$ is a sheaf and A is the presheaf of sections of $\mathbf{A}_0$, then there exists a natural map $\mathbf{A}_0\to\mathbf{A}$ which is a homeomorphism and preserves group operations, where $\mathbf{A}$ is the sheaf generated by A.

Proof. For each α in $\mathbf{A}_0$, $\alpha\in s(\mathrm{U})$ for some section s of $\mathbf{A}_0$ over some open set U. If $\pi(\alpha)=x$, then we define $\mathbf{A}_0\to\mathbf{A}$ by $\alpha\mapsto s_x$. It is welldefined and satisfies the requirements. Q. E. D.

1.2 **Proposition.** If A is a presheaf and $\mathbf{A}$ is the sheaf generated by A. Then for each open set U of X, there is a natural map θ_{U}: $\mathrm{A}(\mathrm{U})\to\mathbf{A}(\mathrm{U})$ which is a homomorphism and commutes with restrictions.

Proof. For g in A(U), we define $\theta_{\mathrm{U}}(g)$: $\mathrm{U}\to\mathbf{A}$ by the assignment $x\mapsto g_x$, then the map θ_{U}: $g\mapsto\theta_{\mathrm{U}}(g)$ is the required homomorphism. Q. E. D.

§2. THE CONSTRUCTIONS OF HOMOLOGY AND COHOMOLOGY GROUPS IN SHEAF

A differential sheaf is a graded sheaf $\{\mathbf{L}^p: p \text{ is an integer}\}$ together with homomorphisms d: $\mathbf{L}^p\to\mathbf{L}^{p+1}$ such that $d^2=0$.

A resolution of a sheaf $\mathbf{A}$ is a differential sheaf $\mathbf{L}^*$ with $\mathbf{L}^p=0$ for $p<0$, together with an "argumentation" homomorphism ε: $\mathbf{A}\to\mathbf{L}^0$ such that the sequence $0\to\mathbf{A}\to\mathbf{L}^0\overset{d}{\to}\mathbf{L}^1\overset{d}{\to}\mathbf{L}^2\to\cdots$ is exact.

Since the exactness of sequences commutes with direct limits, thus we see that the functor assigning to a presheaf its associated sheaf is an exact functor. Thus, if $A\overset{f}{\to}B\overset{g}{\to}C$ is a sequence of presheaves of order two, and if $\mathbf{A}\overset{f'}{\to}\mathbf{B}\overset{g'}{\to}\mathbf{C}$ is the induced sequence of generated sheaves, then $\mathrm{Im}(f')$ and $\mathrm{Ker}(g')$ are generated respectively by the presheaves $\mathrm{Im}(f)$ and $\mathrm{Ker}(g)$. Similarly, the sheaf $\mathrm{Ker}\ g'/\mathrm{Im}\ f'$ is the sheaf generated by the presheaf:

$$U\to \mathrm{Ker}\ g_U/\mathrm{Im}\ f_U.$$

If $\mathbf{L}^*$ is a differential sheaf, we define its homology sheaf (or derived sheaf) to be the graded sheaf $\mathbf{H}^*(\mathbf{L}^*)$ as usual,

$$\mathbf{H}^p(\mathbf{L}^*)=\mathrm{Ker}(d\colon \mathbf{L}^p\to\mathbf{L}^{p+1})/\mathrm{Im}(d\colon \mathbf{L}^{p-1}\to\mathbf{L}^p).$$

By the arguments above, we know that this sheaf is generated by the presheaf

$$U\to H^*(\mathbf{L}^*(U)).$$

For any sheaf $\mathbf{A}$ on X and open set $U\subset X$, let $C^0(U;\mathbf{A})$ be the collection of serration[1], that is $C^0(U;\mathbf{A})=\prod_{x\in U}\mathbf{A}_x$, then the functor $U\to C^0(U;\mathbf{A})$ is a presheaf on X and in this case θ_U is an isomorphism, thus we consider this presheaf as a sheaf, we denote it by $\mathbf{C}^0(U;\mathbf{A})$. And there exists a natural monomorphism ε: $\mathbf{A}\to\mathbf{C}^0(X;\mathbf{A})$ defined by mapping each f_x at x for $f\in\mathbf{A}(U)$ to the germ of the section $y\mapsto f_y$, $y\in U$ at x. If φ is a family of supports on X, we put $C^0_\varphi(X;\mathbf{A})=\Gamma_\varphi(\mathbf{C}^0(X;\mathbf{A}))=\{s\in\mathbf{C}^0(X;\mathbf{A})=C^0(X;\mathbf{A})\colon |s|\in\varphi\}$ and let $\mathbf{Z}^1(X;\mathbf{A})$ be the cokernel of ε so that the sequence: $0\to\mathbf{A}\to\mathbf{C}^0(X;\mathbf{A})\to\mathbf{Z}^1(X;\mathbf{A})\to 0$ is exact. We also define inductively $\mathbf{C}^n(X;\mathbf{A})=\mathbf{C}^0(X;\mathbf{Z}^n(X;\mathbf{A}))$, $\mathbf{Z}^{n+1}(X;\mathbf{A})=\mathbf{Z}^1(X;\mathbf{Z}^n(X;\mathbf{A}))$. Thus each sequence $0\to\mathbf{Z}^n(X;\mathbf{A})\overset{\varepsilon}{\to}\mathbf{C}^n(X;\mathbf{A})\overset{\partial}{\to}\mathbf{Z}^{n+1}(X;\mathbf{A})\to 0$ is exact. Let $d=\varepsilon\cdot\partial$ be the composition $\mathbf{C}^n(X;\mathbf{A})\overset{\partial}{\to}\mathbf{Z}^{n+1}(X;\mathbf{A})\overset{\varepsilon}{\to}\mathbf{C}^{n+1}(X;\mathbf{A})$, then the sequence $0\to\mathbf{A}\to\mathbf{C}^0(X;\mathbf{A})\overset{d}{\to}\mathbf{C}^1(X;\mathbf{A})\overset{d}{\to}\mathbf{C}^2(X;\mathbf{A})\to\cdots$ is exact. It is called the

canonical resolution of $\mathbf{A}$. This resolution is pointwise homotopically trivial.

For a family of supports on X, (which means that φ is a family of closed subsets of X such that a closed subset of a member of φ is in φ and φ is closed under finite union), we put $C^n_\varphi(X; \mathbf{A})=\Gamma_\varphi(\mathbf{C}^n(X; \mathbf{A}))=\mathbf{C}^0_\varphi(X; \mathbf{Z}^n(X; \mathbf{A}))$, which is an exact functor. Finally we define $H^n_\varphi(X; \mathbf{A})=H^n(C^*_\varphi(X; \mathbf{A}))$. They form the sheaf cohomology groups.

We have following results:

(1) Since $0\to\Gamma_\varphi(\mathbf{A})\to\Gamma_\varphi(\mathbf{C}^0(X; \mathbf{A}))\to\Gamma_\varphi(\mathbf{C}^1(X; \mathbf{A}))$ is exact, thus $\Gamma_\varphi(\mathbf{A})=H^0_\varphi(X; \mathbf{A})$.

(2) Given a short exact sequence $0\to\mathbf{A}'\to\mathbf{A}\to\mathbf{A}''\to0$ of sheaves on X, we have a short exact sequence $0\to C^*_\varphi(X; \mathbf{A}')\to C^*_\varphi(X; \mathbf{A})\to C^*_\varphi(X; \mathbf{A}'')\to0$ of chain complexes and thus an induced long exact sequence:

$$\cdots\to H^p_\varphi(X; \mathbf{A}')\to H^p_\varphi(X; \mathbf{A})\to H^p_\varphi(X; \mathbf{A}'')\xrightarrow{\delta} H^{p+1}_\varphi(X; \mathbf{A}')\to\cdots.$$

§ 3. ACYCLIC SHEAVES AND FLABBY SHEAVES

A sheaf $\mathbf{A}$ is said to be φ-acyclic if $H^p_\varphi(X; \mathbf{A})=0$ for $p>0$.

3.1 **Theorem.** If $\mathbf{L}^*$ is a resolution of $\mathbf{A}$ by φ-acyclic sheaves, then there exists a map r: $H^p(\Gamma_\varphi(\mathbf{L}^*))\to H^p_\varphi(X; \mathbf{A})$ which is an isomorphism for all p.

Proof. Let $\mathbf{Z}^p=\mathrm{Ker}(\mathbf{L}^p\to\mathbf{L}^{p+1})=\mathrm{Im}((\mathbf{L}^{p-1}\to\mathbf{L}^p)$, where $\mathbf{Z}^0=\mathbf{A}$. Then the exact sequence $0\to\mathbf{Z}^{p-1}\to\mathbf{L}^{p-1}\to\mathbf{Z}^p\to0$ induces the exact sequence $0\to\Gamma_\varphi(\mathbf{Z}^{p-1})\to\Gamma_\varphi(\mathbf{L}^{p-1})\to\Gamma_\varphi(\mathbf{Z}^p)\to H^1_\varphi(X; \mathbf{Z}^{p-1})$. Thus, we have the monomorphism:

(1) $H^p(\Gamma_\varphi(\mathbf{L}^*))=\dfrac{\Gamma_\varphi(\mathbf{Z}^p)}{\mathrm{Im}(\Gamma_\varphi(\mathbf{L}^{p-1})\to\Gamma_\varphi(\mathbf{L}^p))}\to H^1_\varphi(X; \mathbf{Z}^{p-1})$. Moreover, the sequence $0\to\mathbf{Z}^{p-r}\to\mathbf{L}^{p-r}\to\mathbf{Z}^{p-r+1}\to0$ induces a homomorphism:

(2) $H^{r-1}_\varphi(X; \mathbf{Z}^{p-r+1})\to H^r_\varphi(X; \mathbf{Z}^{p-r})$.

And let r be the composition:

$$H^p(\Gamma_\varphi(\mathbf{L}^*))\to H^1_\varphi(X; \mathbf{Z}^{p-1})\to H^2_\varphi(X; \mathbf{Z}^{p-2})\to\cdots\to H^p_\varphi(X; \mathbf{Z}^0)=H^p_\varphi(X; \mathbf{A}).$$

If φ is acyclic, then (1) and (2) are isomorphism, so is r. Q. E. D.

A sheaf $\mathbf{A}$ on X is said to be flabby if $\mathbf{A}(X)\to\mathbf{A}(U)$ is onto for every open set $U\subset X$.

3.2 **Proposition.** If f: X→Y is continuous and **A** is a flabby sheaf on X, then $f\mathbf{A}$ is flabby (where $f\mathbf{A}$ denote the sheaf generated by the presheaf U→$\mathbf{A}(f^{-1}(U))$ on Y).

Proof. Since each θ_U is an isomorphism, thus $f\mathbf{A}(U)=\mathbf{A}(f^{-1}(U))$. Therefore the flabby property of $f\mathbf{A}$ follows from the flabby property of **A.**

3.3 **Theorem.** Let $0\to\mathbf{A}'\xrightarrow{h'}\mathbf{A}\xrightarrow{h}\mathbf{A}''\to 0$ be exact and suppose that $\mathbf{A}'$ is flabby. Then for any family φ of supports on X, $0\to\Gamma_\varphi(\mathbf{A}')\to\Gamma_\varphi(\mathbf{A})\to\Gamma_\varphi(\mathbf{A}'')\to 0$ is exact. In particular, since $\mathbf{A}'|U$ is flabby for any U open in X, $0\to\mathbf{A}'(U)\to\mathbf{A}(U)\to\mathbf{A}''(U)\to 0$ is exact. If both $\mathbf{A}'$ and **A** are flabby, then $\mathbf{A}''$ is flabby.

Proof. Let $s\in\Gamma_\varphi(\mathbf{A}'')$ and $\mathbf{C}=\{(U,t)\,|\,U$: open in X, $t\in\mathbf{A}(U)$ with $ht(x)=s(x)\}$. At first, we show that $\mathbf{C}\neq\phi$: For a fixed $x_0\in X$, $s(x_0)\in\mathbf{A}''$, there exists an $a_{x_0}\in\mathbf{A}$ with $h(a_{x_0})=s(x_0)$. And a_{x_0} is in the image of some $t\in\mathbf{A}(U)$. Consider the set $U=\{x\,|\,ht(x)=s(x)\}$ which is an open set. Thus $(U,t)\in\mathbf{C}$. Order **C** by $(U,t)<(U',t')$ if $U\subset U'$ and $t'|U=t$, then by Zorn's lemma, there exists a maximal element (V,t) of **C**, and suppose that $V\neq X$. Let $x\notin V$, then by the same way as above, there exists a neighborhood W of x such that $(W,t')\in\mathbf{C}$ for some $t'\in\mathbf{A}(W)$.

Now, $t|V\cap W-t'|V\cap W\in\mathbf{A}'(V\cap W)$ and hence extends to $t''\in\mathbf{A}'(W)$, $t|V\cap W=(t'+t'')|V\cap W$, so t and $t'+t''$ define an element of $\mathbf{A}(V\cup W)$ extending t and representing s on $V\cup W$. This contradicts the maximality of (V,t) and shows that $V=X$.

Let $U=X-|s|$ and $t|U\in\mathbf{A}'(U)$. Thus $t|U$ can be extended to $t'\in\mathbf{A}'(X)$, $t-t'$ represents s on X and is zero on U. Thus $|t-t'|\subset|s|\in\varphi$.

The last statement follows from the commutative diagram with both rows and left column exact:

$$\begin{array}{ccccc}\mathbf{A}(X) & \to & \mathbf{A}''(X) & \to & 0\\ \downarrow & & \downarrow & & \\ \mathbf{A}(U) & \to & \mathbf{A}''(U) & \to & 0\\ \downarrow & & & & \\ 0 & & & & \end{array}$$

Q. E. D.

3.4 **Theorem.** A flabby sheaf is φ-acyclic for any φ.

Proof. Since $\mathbf{C}^0(X;\mathbf{A})$ is flabby, it follows from the above theorem that $\mathbf{Z}^1(X;\mathbf{A})$ is flabby when **A** is flabby. By induction all

$\mathbf{Z}^n(X; \mathbf{A})$ are flabby. Thus if we apply the functor Γ_φ to the exact sequences:

$$0 \to \mathbf{Z}^n(X; \mathbf{A}) \to \mathbf{C}^n(X; \mathbf{A}) \to \mathbf{Z}^{n+1}(X; \mathbf{A}) \to 0, \quad n = 1, 2, \cdots$$

we obtain exact sequences:

$$0 \to \Gamma_\varphi(\mathbf{Z}^n(X; \mathbf{A})) \to C^n_\varphi(X; \mathbf{A}) \to \Gamma_\varphi(\mathbf{Z}^{n+1}(X; \mathbf{A})) \to 0.$$

It follows that the sequence $0 \to \Gamma_\varphi(\mathbf{A}) \to C^0_\varphi(X; \mathbf{A}) \to C^1_\varphi(X; \mathbf{A}) \to \cdots$ is exact, thus $H^p_\varphi(X; \mathbf{A}) = 0$ for $p > 0$. Q. E. D.

§ 4. φ-SOFT SHEAVES

A sheaf $\mathbf{A}$ on X is said to be φ-soft if $\mathbf{A}(X) \to \mathbf{A}(K)$ is onto for all $K \in \varphi$. If φ consists of all closed sets, then $\mathbf{A}$ is said to be soft.

If $Y \subset X$ and φ is a family of supports on X, let $\varphi \cap Y$ denote the family $\{K \cap Y: K \in \varphi\}$ of supports on Y, and $\varphi | Y$ denote the family $\{K: K \subset Y \text{ and } K \in \varphi\}$ of supports on Y. A support family φ on X is said to be paracompactifying if each element of φ is paracompact, and each element of φ has a neighborhood which is in φ.

4.1 **Proposition.** Let φ be paracompactifying. Then the following statements are equivalent:

(1) $\mathbf{A}$ is φ-soft,

(2) $\mathbf{A}|K$ is soft for every $K \in \varphi$,

(3) $\Gamma_\varphi(\mathbf{A}) \to \Gamma_{\varphi|F}(\mathbf{A}|F)$ is onto for all closed F in X.

Proof. (3)$\Rightarrow$(1); It is trivial.

(1)$\Rightarrow$(2); Let $K \in \varphi$ and $K' \in \varphi$ with $K' \subset K$. Since $\mathbf{A}$ is φ-soft, we have that $\mathbf{A}(X) \to \mathbf{A}(K')$ is onto, that is for each section $s: K' \to \mathbf{A}$, there exists a section $s': X \to \mathbf{A}$ which is an extension of s. Thus $s'|K: K \to \mathbf{A}$ is an extension of s to K, therefore $\mathbf{A}(K) \to \mathbf{A}(K')$ is onto.

(2)$\Rightarrow$(3); Let $s \in \Gamma_{\varphi|F}(\mathbf{A}|F)$, $K = |s|$ and let $K' \in \varphi$ be a neighborhood of K, B be the boundary of K'. By (2), the element of $\mathbf{A}((K' \cap F) \cup B)$ which is s on $K' \cap F$ and zero on B can be extended to $s' \in \mathbf{A}(K')$. Clearly s' extends, by zero, to $s'' \in \mathbf{A}(X)$ with $|s''| \subset K'$, s'' is the desired extension of s. Q. E. D.

4.2 **Theorem.** Let A be a subspace of X having a fundamental

system of paracompact neighborhoods. Then, for any sheaf $\mathbf{A}$ on X, $\mathbf{A}(A)=\varinjlim \mathbf{A}(U)$ where U ranges ever the neighborhoods of A.

Proof. Let $s\in\mathbf{A}(A)$ and let π be the local homeomorphism $\mathbf{A}\to X$. Then for each $a\in X$, there exists an open set U'_a in X containing a and open set N_a in $\mathbf{A}$ containing $s(a)$, they are homeomorphic under $\pi^{-1}|U'_a$. Since $s(U'_a\cap A)\cap N_a$ is open in $s(U'_a\cap A)$ and $s|U'_a\cap A$ is continuous, thus there exists an open set U_a in X such that U_a contains a and $s(U_a\cap A)\subset s(U'_a\cap A)\cap N_a$. Let $s_a=\pi^{-1}|U_a$. Therefore, we have $s_a\in\mathbf{A}(U_a)$ with $s_a|U_a\cap A=s|U_a\cap A$. And thus we obtain a covering of A by those open sets U_a.

We may assume that X is paracompact and that U_a cover X; otherwise, since $A\subset\bigcup U_a$, there exists a paracompact neighborhood Y of A with $A\subset Y\subset\bigcup U_a$, we work in Y. Thus we may also assume that $\{U_a\}$ is locally finite. Since the paracompact space is normal, there exists an open covering $\{V_a\}$ of X with $\bar{V}_a\subset U_a$ for all α. Let $W=\{x\in X:\ x\in\bar{V}_a\cap\bar{V}_\beta\Rightarrow s_a(x)=s_\beta(x)\}$, $J(x)=\{\alpha:\ x\in\bar{V}_a\}$. Then $J(x)$ is finite and every $x\in X$ has a neighborhood $N(x)$ such that $y\in N(x)\Rightarrow J(y)\subset J(x)$.

If $x\in W$, the sections s_a for $\alpha\in J(x)$, coincide in a neighborhood of x. Thus W is open. Also $A\subset W$. Now let $t\in\mathbf{A}(W)$ be defined by $t(x)=s_a(x)$ when $x\in V_a\cap W$; t then is well-defined by the definition of W, and is continuous, since it coincides with s_a on $V_a\cap W$. Thus t extends s. Q. E. D.

4.3 **Corollary.** If φ is paracompactifying, then every flabby sheaf on X is φ-soft.

Proof. For each $K\in\varphi$, there exists a paracompact neighborhood K' of K, $K'\in\varphi$. Since K is normal, thus for each open set U containing K, there exists an open set V with $K\subset V\subset\bar{V}\subset U$. Then $K\subset\bar{V}\cap K'\subset U$, where $\bar{V}\cap K'$ is paracompact. That is we have proved that K has a fundamental system of paracompact neighborhoods. Let $s\in\mathbf{A}(K)$, then by the theorem above, there exists an open neighborhood W of K and s can be extended to a section s' over W. Since $\mathbf{A}$ is flabby, s' can be extended to section s'' over X. Therefore $\mathbf{A}(X)\to\mathbf{A}(K)$ is onto. Q. E. D.

4.4 **Theorem.** Let φ be paracompactifying and suppose that $0\to\mathbf{A}'\to\mathbf{A}\to\mathbf{A}''\to 0$ is exact with $\mathbf{A}'$ φ-soft. Then the sequence $0\to\Gamma_\varphi(\mathbf{A}')\to\Gamma_\varphi(\mathbf{A})\to\Gamma_\varphi(\mathbf{A}'')\to 0$ is exact.

Proof. We need only to show that $\Gamma_\varphi(\mathbf{A})\to\Gamma_\varphi(\mathbf{A}'')$ is onto.

Let $s\in\Gamma_\varphi(\mathbf{A}'')$, let $\mathrm{K}=|s|\in\varphi$, and let $\mathrm{K}'\in\varphi$ be a neighborhood of K. Suppose that we can find an element $t\in\mathbf{A}(\mathrm{K}')$ representing $s|\mathrm{K}'$. Then, on the boundary B of K', $t|\mathrm{B}\in\mathbf{A}'(\mathrm{B})$ can be extended to $\mathbf{A}'(\mathrm{K}')$. Subtracting this from t, we see that we may suppose $t|\mathrm{B}=0$. But then t can be extended by zero to X. Thus we see that we may as well assume that X is paracompact, and φ is the class of all closed sets.

Since the homomorphism between sheaves is also a local homeomorphism, and by above assumptions, if $s\in\mathbf{A}''(\mathrm{X})$, then there exists a locally finite covering $\{\mathrm{U}_\alpha\}$ of X with $s_\alpha\in\mathbf{A}(\mathrm{U})$ representing $s|\mathrm{U}_\alpha$. Let $\{\mathrm{V}_\alpha\}$ be a covering of X with $\bar{\mathrm{V}}_\alpha\subset\mathrm{U}$.

We order the index set $\{\alpha\}$, and put $\mathrm{F}_\alpha=\bigcup_{\beta<\alpha}\bar{\mathrm{V}}_\beta$. Since $\{\mathrm{U}_\alpha\}$ is locally finite, F_α is closed for all α. We shall define inductively an element $t_\alpha\in\mathbf{A}(\mathrm{F}_\alpha)$ representing $s|\mathrm{F}_\alpha$, such that $t_\alpha|\mathrm{F}_\beta=t_\beta$ for all $\beta<\alpha$. Suppose that t_β has been defined for all $\beta<\alpha$. If α is a limit ordinal, let $\mathrm{F}_\alpha=\bigcup_{\beta<\alpha}\mathrm{F}_\beta$; then t_α is defined in the natural way. If α is the successor of α', then both $t_{\alpha'}$ and $s_{\alpha'}$ represent s on $\mathrm{F}_{\alpha'}\cap\bar{\mathrm{V}}_{\alpha'}$. The difference is a section of $\mathbf{A}'$ over $\mathrm{F}_{\alpha'}\cap\bar{\mathrm{V}}_{\alpha'}$ and hence can be extended to $\mathbf{A}'(\bar{\mathrm{V}}_{\alpha'})$. Thus $t_{\alpha'}$ can be extended to F_α representing $s|\mathrm{F}_\alpha$. Q. E. D.

4.5 **Proposition.** If φ is paracompactifying and $0\to\mathbf{A}'\to\mathbf{A}\to\mathbf{A}''\to 0$ is exact with $\mathbf{A}'$ and A φ-soft, then $\mathbf{A}''$ is also φ-soft.

Proof. Let $\mathrm{K}\in\varphi$, we consider every $\mathrm{K}'\in\varphi$ with $\mathrm{K}'\subset\mathrm{K}$. In the diagram:

$$\begin{array}{ccccc}
\mathbf{A}(\mathrm{K}) & \to & \mathbf{A}''(\mathrm{K}) & \to & 0 \\
\downarrow & & \downarrow & & \\
\mathbf{A}(\mathrm{K}') & \to & \mathbf{A}''(\mathrm{K}') & \to & 0 \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

Since $\mathbf{A}'$, $\mathbf{A}$ are φ-soft, by Proposition 4.1, $\mathbf{A}$, $\mathbf{A}'$ are $\varphi|\mathrm{K}$-soft. Thus the top and bottom rows and the left column are exact. It follows that $\mathbf{A}''(\mathrm{K})\to\mathbf{A}''(\mathrm{K}')$ is onto. That is $\mathbf{A}''$ is $\varphi|\mathrm{K}$-soft. Thus $\mathbf{A}''$ is φ-soft. Q. E. D.

4.6 **Theorem.** A φ-soft sheaf is φ-acyclic if φ is paracompactifying.

Proof. Consider the sequence $0\to\mathbf{A}\to\mathbf{C}^0(X;\mathbf{A})\to\mathbf{Z}^1(X;\mathbf{A})\to 0$, $\mathbf{A}$ is φ-soft. Since $\mathbf{C}^0(X;\mathbf{A})$ is flaby, it follows from the associated exact cohomology sequence: $0=H_\varphi^{n-1}(X;\ \mathbf{C}^0(X;\ \mathbf{A}))\to H_\varphi^{n-1}(X;\ \mathbf{Z}^1(X;\ \mathbf{A}))\to H_\varphi^n(X;\mathbf{A})\to H_\varphi^n(X;\mathbf{C}^0)=0$ for $n>1$; we have $H_\varphi^n(X;\mathbf{A})=H_\varphi^{n-1}(X;\mathbf{Z}^1(X;\mathbf{A}))$ for $n>1$.

Since $\mathbf{A}$ is φ-soft. Thus $0\to\Gamma_\varphi(\mathbf{A})\to\Gamma_\varphi(\mathbf{C}^0)\to\Gamma_\varphi(\mathbf{Z}^1)\to 0$ is exact. Thus $H_\varphi^1(X;\mathbf{A})=0$, and then $H_\varphi^k(X;\mathbf{Z}^1(X;\mathbf{A}))=0$ for $k>0$. By Corollary 4.3, $\mathbf{Z}^1(X;\mathbf{A})$ is also φ-soft. So that the theorem follows by induction. Q. E. D.

A subspace A of a space X is said to be relatively Hausdorff if any two points of A have disjoint open neighborhoods in X.

4.7 **Theorem.** Let A be a compact relatively Hausdorff subset of a space X. Then for any sheaf $\mathbf{A}$ on X, $\mathbf{A}(A)=\varinjlim \mathbf{A}(U)$, where U ranges over the neighborhoods of A in X.

Proof. Let $s\in\mathbf{A}(A)$. We follow the same procedure as in proving Theorem 4.2. Since A is compact, we can obtain a finite open covering $\{U_i\}$ of A in X and elements $s_i\in\mathbf{A}(U_i)$ with $s_i|A\cap U_i=s|A\cap U_i$. Next, we shrink the open covering $\{U_i\cap A\}$ of A to get compact sets $K_i\subset U_i\cap A$ with $A=\cup K_i$. Thus it suffices to prove the following assertion (and by induction):

If P_1, P_2, are compact subsets of A with open neighborhoods V_1, V_2 resp. in X, and if $t_i\in\mathbf{A}(V_i)$ coincide on $P_1\cap P_2$, then there is a section t over some neighborhood of $P_1\cup P_2$ coinciding with t_i on P_i, $i=1,2$.

Since t_1 and t_2 coincide on some neighborhood V of $P_1\cap P_2$, we may suppose that $V\subset V_1\cap V_2$. The sets P_1-V, P_2-V are disjoint compact, since A is relatively Hausdorff, they have disjoint open neighborhoods $Q_i\supset P_i-V$ in X. The sections $t_1|Q_1$, $t_1|V=t_2|V$ and $t_2|Q_2$ coincide on their common domain and thereby provide a section t on $Q_1\cup V\cup Q_2\supset P_1\cup P_2$. Q. E. D.

4.8 **Corollary.** If A is a compact relatively Hausdorff subspace of X, then $\mathbf{A}|A$ is soft for any flabby sheaf $\mathbf{A}$ on X.

Proof. For each closed subset A' of A, A' is also a compact relatively Hausdorff subspace of X. Let $t \in \mathbf{A}(A')$. By Theorem 4.7, t can be extended to an open neighborhood U of A'. Since $\mathbf{A}$ is flabby, this section can again be extended to X. Then restricting it on A, we get the required extension of t to A. Q. E. D.

§ 5. THE CONSTRUCTION OF RELATIVE COHOMOLOGY

Let $A \subset X$, and let φ be a family of supports on X. For any sheaf $\mathbf{A}$ on X, we have the natural homomorphism $\mathbf{C}^*(X; \mathbf{A}) \to \mathbf{C}^*(A; \mathbf{A}|A)$, or equivalently the homomorphism i^*: $\mathbf{C}^*(X; \mathbf{A}) \to i\,\mathbf{C}^*(A; \mathbf{A}|A)$ of sheaves on X.

Let $\operatorname{Ker} i^* = \mathbf{C}^*(X, A; \mathbf{A})$, $C^*_\varphi(X, A; \mathbf{A}) = \Gamma_\varphi(\mathbf{C}^*(X, A; \mathbf{A}))$, and then define $H^*_\varphi(X, A; \mathbf{A}) = H^*(C^*_\varphi(X, A; \mathbf{A}))$ as the relative cohomology.

Since $\operatorname{Ker} i$ is flabby and $\Gamma_\varphi(i\beta) = \Gamma_{\varphi \cap A}(\mathbf{B})$, thus the exact sequence: $0 \to \operatorname{Ker} i^* \to \mathbf{C}^*(X; \mathbf{A}) \to i\,\mathbf{C}^*(A; \mathbf{A}|A) \to 0$ induces an exact sequence: $0 \to \Gamma_\varphi(\operatorname{Ker} i^*) \to \Gamma_\varphi(\mathbf{C}^*(X; \mathbf{A})) \to \Gamma_\varphi(i\,\mathbf{C}^*(A; \mathbf{A}|\mathbf{A})) \to 0$. That is $0 \to C^*_\varphi(X, A; \mathbf{A}) \to C^*_\varphi(X; \mathbf{A}) \to C^*_{\varphi \cap A}(A; \mathbf{A}|A) \to 0$, and hence a long exact sequence: $\cdots \to H^p_\varphi(X, A; \mathbf{A}) \to H^p_\varphi(X; \mathbf{A}) \to H^p_{\varphi \cap A}(A; \mathbf{A}|A) \to H^{p+1}_\varphi(X, A; \mathbf{A}) \to \cdots$ And here $\mathbf{C}^*(X, A; \mathbf{A})$ and $C^*_\varphi(X, A; \mathbf{A})$ are exact functors of $\mathbf{A}$.

§ 6. DIMENSION

Let φ be a family of supports on X and let L be a fixed ground ring with unit (or, more generally, a sheaf of rings). We let $\dim_{\varphi, L} X$ be the smallest integer n (or ∞) such that $H^k_\varphi(X; \mathbf{A}) = 0$ for all sheaves $\mathbf{A}$ of L-modules and all $k > n$.

6.1 **Proposition.** The following statements are equivalent:

(1) $\dim_{\varphi, L} X \leq n$,

(2) For any sheaf $\mathbf{A}$ of L-modules, $\mathbf{Z}^n(X; \mathbf{A})$ is φ-acyclic,

(3) Every sheaf $\mathbf{A}$ of L-modules has a φ-acyclic resolution of length n.

Proof. (1)⇒(2): Let $\mathbf{Z}^p$ denote $\mathbf{Z}^p(X; \mathbf{A})$ and $\mathbf{C}^p$ denote $\mathbf{C}^p(X; \mathbf{A})$.

We have exact sequences $0 \to \mathbf{Z}^p \to \mathbf{C}^p \to \mathbf{Z}^{p+1} \to 0$ for all $p \geq 0$, then $H^k_\varphi(X; \mathbf{Z}^n) \cong H^{k+1}_\varphi(X; \mathbf{Z}^{n-1}) \cong \cdots \cong H^{n+k-1}_\varphi(X; \mathbf{Z}^1) \cong H^{n+k}_\varphi(X; \mathbf{A}) = 0$ for any $k > 0$.

(2)⇒(3): $0\to\mathbf{A}\to\mathbf{C}^0\to\mathbf{C}^1\to\cdots\to\mathbf{C}^{n-1}\to\mathbf{Z}^n\to 0$ is a φ-acyclic resolution of length n.

(3)⇒(1): Let $0\to\mathbf{A}\to\mathbf{L}^0\to\mathbf{L}^1\to\cdots\to\mathbf{L}^n\to 0$ be a φ-acyclic resolution of $\mathbf{A}$ of length n and $\bar{\mathbf{Z}}^p=\mathrm{Ker}(\mathbf{L}^p\to\mathbf{L}^{p+1})$. Then for $k>0$, we have that:

$$H_\varphi^{n+k}(X;\mathbf{A})\cong H_\varphi^{n+k-1}(X;\bar{\mathbf{Z}}^1)\cong\cdots\cong H_\varphi^k(X;\bar{\mathbf{Z}}^n)\cong H_\varphi^k(X;\mathbf{L}^n)=0.$$

Q. E. D.

§ 7. SPECTRAL SEQUENCES OF DOUBLE COMPLEXES

Suppose that $C^{*,*}$ is a double complex over some fixed ring, that is a family of modules doubly indexed by the integers and with differentials d': $C^{p,q}\to C^{p+1,q}$ and d'': $C^{p,q}\to C^{p,q+1}$ such that $(d')^2=0$, $(d'')^2=0$ and $d'd''+d''d'=0$.

Let C^* be the total complex with $C^n=\sum_{p+q=n}C^{p,q}$ and the differential $d=d'+d''$. We have two filtrations in the complex C^*: the first filtration ${}'F$ is defined by ${}'F_tC^n=\sum_{p\geq t}C^{p,q}$, $p+q=n$; the second filtration ${}''F$ is defined by ${}''F_tC^n=\sum_{q\geq t}C^{p,q}$, $p+q=n$.

From these filtrations, we obtain two spectral sequences[2] denoted by $\{{}'E_r^{p,q},d_r'\}$ and $\{{}''E_r^{p,q},d_r''\}$ respectively where d_r': ${}'E_r^{p,q}\to{}'E_r^{p+r,q-r+1}$ and d_r'': ${}''E_r^{p,q}\to{}''E_r^{p+r,q-r+1}$. In these notations, p is the filtration degree and q is the complementary degree. We denote the homology with respect to d' and d'' by ${}'H$ and ${}''H$ respectively. Then we have that: ${}'E_1^{p,q}={}''H^q(C^{p,*})$ and ${}''E_1^{p,q}={}'H^q(C^{*,p})$. And we have the differential operators d_1' on ${}'E_1$ terms and d_1'' on ${}''E_1$ terms which are induced by d' and d'' respectively. Therefore:

$${}'E_2^{p,q}={}'H^p({}''H^q(C^{*,*}))\text{ and }{}''E_2^{p,q}={}''H^p({}'H^q(C^{*,*})).$$

If $C^{p,q}=0$ for $p<p_0$ (p_0 is fixed), then ${}''F_tC^n=0$ for $t>n-p_0$. Here we say that the second filtration is regular. While if $C^{p,q}=0$ for $q<q_0$, then ${}'F_tC^n=0$ for $t>n-q_0$, we say that the first filtration is regular. Thus we have another useful condition implying the regularities of both filtrations, namely that there exist integers p_0 and p_1 such that $C^{p,q}=0$ for $p<p_0$ and for $p>p_1$.

Now, we introduce two decreasing filtrations in $H^*(C^*)$:

${}'F_pH^n(C^*)=\mathrm{Im}(H^n({}'F_pC^*)\to H^n(C^*))$, ${}''F_pH^n(C^*)=\mathrm{Im}(H^n({}''F_pC^*)\to H^n(C^*))$. Two graded modules associated with these two filtrations ${}'F_pH^n(C^*)$ and ${}''F_pH^n(C^*)$ are defined to be ${}'G_pH^n(C^*)$ and ${}''G_pH^n(C_*)$ respectively, where ${}'G_pH^n(C^*)={}'F_pH^n(C^*)/{}'F_{p+1}H^n(C^*)$ and ${}''G_pH^n(C^*)={}''F_pH^n(C^*)/{}''F_{p+1}H^n(C^*)$. When both filtrations are regular, we have that ${}'E_\infty^{p,q}\cong{}'G_pH^{p+q}(C^*)$ and ${}''E^{p,q}\cong{}''G_pH^{p+q}(C^*)$.

§ 8. THE SPECTRAL SEQUENCE OF A DIFFERENTIAL SHEAF

Let $\mathbf{L}^*$ be a differential sheaf. We consider two cases:

(a) $\mathbf{L}^q=0$ for $q<q_0$

(b) $\dim_\varphi X<\infty$ with respect to a given ground ring and family φ of supports.

In case (a), let $L^{*,*}=\sum L^{p,q}$, where $L^{p,q}=C_\varphi^p(X;\mathbf{L}^q)$ and $(-1)^p d''$: $L^{p,q}\to L^{p,q+1}$ be the differential induced by the homomorphism $\mathbf{L}^q\to\mathbf{L}^{q+1}$ and d': $L^{q,p}\to L^{p+1,q}$ be the differential of the complex $C_\varphi^*(X;\mathbf{L}^q)$. Let $d=d'+d''$ be the total differential and L be the total complex, where $L^n=\sum_{p+q=n}L^{p,q}$.

There are two spectral sequences ${}'E^{p,q}$ and ${}''E_r^{p,q}$ of the double complex $L^{*,*}$ converging to the graded groups associated with filtrations on $H^{p+q}(L^*)$. We abbreviate this statement by the notation $E_2^{p,q}\Rightarrow H^{p+q}(L^*)$.

We have ${}'E_2^{p,q}={}'H^p({}''H^q(L^{*,*}))$ and ${}''E_2^{p,q}={}''H^p({}'H^q(L^{*,*}))$, where ${}'H$ and ${}''H$ are computed by using d' and d'' respectively. Since C_φ^* is an exact functor, we have that ${}''H^q(L^{*,*})=C_\varphi^*(X;\mathbf{H}^q(\mathbf{L}^*))$. Thus ${}'E_2^{p,q}=H_\varphi^p(X;\mathbf{H}^q(\mathbf{L}^*))$ and ${}'H^q(L^{*,*})=H_\varphi^q(X;\mathbf{L}^*)$, thus ${}''E_2^{p,q}=H^p(H_\varphi^q(X;\mathbf{L}^*))$.

If we are in case (b) with $\dim_\varphi X\leq n$, then $0\to\mathbf{A}\to\mathbf{C}^0(X;\mathbf{A})\to\cdots\to\mathbf{C}^{n-1}(X;\mathbf{A})\to\mathbf{Z}^n(X;\mathbf{A})\to 0$ is a resolution of $\mathbf{A}$ by φ-acyclic sheaves and is an exact functor of $\mathbf{A}$. In the discussion above, $C_\varphi^*(X;\mathbf{A})$ should be replaced by Γ_φ of this resolution, and then by Theorem 3.1, the spectral sequences are not changed from E_2 on.

8.1. **Theorem.** Let $\mathbf{L}^*$ be a differential sheaf. Assume that either $\dim_\varphi X<\infty$ or that $\mathbf{L}^q=0$ for $q<q_0$. Also assume that $H^*(H_\varphi^q(X;\mathbf{L}^*))=0$ for $q>0$. Then there is a spectral sequence with $E_2^{p,q}=H_\varphi^p(X;\mathbf{H}^q(\mathbf{L}^*))\Rightarrow H^{p+q}(\Gamma_\varphi(\mathbf{L}^*))$.

Proof. Since we have ${}''E_2^{p,q}=H^p(H_\varphi^q(X;\mathbf{L}^*))=0$ for $q>0$. Thus $E_\infty^{p,q}=0$ for $q\neq 0$

And from:
$$0=\left\{\begin{array}{ccccc} {}''E_2^{p-2,1} & \to & {}''E_2^{p,0} & \to & {}''E_2^{p+2,-1} \\ \| & & & & \\ {}''E_3^{p-3,2} & \to & {}''E_3^{p,0} & \to & {}''E_3^{p+3,-2} \\ \| & & & & \\ \vdots & & & & \\ \| & & & & \\ {}''E_\infty^{p,0} & & & & \end{array}\right\}=0\ .$$

We have that: ${}''E_\infty^{p,0}={}''E_2^{p,0}=H^p(\Gamma_\varphi(\mathbf{L}^*))$. And here ${}''E_\infty^{p,0}=\sum_q {}''E_\infty^{q,p-q}$ $=\sum_q {}''F_qH^p(L^*)/{}''F_{q+1}H^p(L^*)\cong H^p(L^*)$. Therefore $H^p(\Gamma_\varphi(\mathbf{L}^*))\cong H^p(L^*)$.

Since $\mathbf{L}^q=0$ for $q<q_0$ or $\dim_\varphi X<\infty$, thus the first filtration is regular. Thus we have that ${}'E_2^{p,q}=H_\varphi^p(X;\mathbf{H}^q(\mathbf{L}^*))\Rightarrow H^{p+q}(L^*)$, that is ${}'E_2^{p,q}\Rightarrow H^{p+q}(\Gamma_\varphi(\mathbf{L}^*))$. Q. E. D.

8.2. **Theorem.** Let $\mathbf{L}^*$ be a differential sheaf such that $\mathbf{H}^q(\mathbf{L}^*)=0$ for $q\neq 0$. suppose either that $\dim_\varphi X<\infty$ or that $\mathbf{L}^q=0$ for $q<q_0$. Then, with $\mathbf{L}=\mathbf{H}^0(\mathbf{L}^*)$, there is a spectral sequence such that:

$$E_2^{p,q}=H^p(H_\varphi^q(X;\mathbf{L}^*))\Rightarrow H_\varphi^{p+q}(X;\mathbf{L}).$$

Proof. We have that: $${}'E_2^{p,q}=\left\{\begin{array}{ll} H_\varphi^p(X;\mathbf{L}) & q=0 \\ 0 & q\neq 0. \end{array}\right.$$

It follows that $H_\varphi^p(X;\mathbf{L})$ and $H^p(L^*)$ are isomorphic (the proof is the same as in Theorem 8.1). And since the second filtration is regular, thus we have that: ${}''E_2^{p,q}=H^p(H_\varphi^q(X;\mathbf{L}^*))\Rightarrow H_\varphi^{p+q}(X;\mathbf{L})$.

Q. E. D.

§ 9. THE LERAY SHEAF

Let f: $X\to Y$ be continuous and let ψ be a family of supports on X. For $U\subset Y$, we let $U^*=f^{-1}(U)$ and let $\psi(U)=\{A\,|\,A$: closed subset of U^*, and for any y in U, there is a neighborhood $N(y)$ such that $A\cap N^*\in\psi\cap N^*\}$.

Furthermore, if $\mathbf{A}$ is a sheaf on X, we consider the presheaf $U\to\Gamma_{\psi(U)}(\mathbf{A}|U^*)$ on Y, which is a sheaf. And this sheaf is called the direct image of $\mathbf{A}$ with respect to ψ and is denoted by $f_\psi\mathbf{A}$.

For $\psi \cap U^* \subset \psi(U)$, so that there is a natural map: $\Gamma_{\varphi\cap U^*}(\mathbf{A}|U^*) \to \Gamma_{\psi(U)}(\mathbf{A}|U^*) = (f_\psi \mathbf{A})(U)$. And this map induces an isomorphism of the generated sheaves. Therefore f is the sheaf generated by the presheaf $U \to \Gamma_{\psi\cap U^*}(\mathbf{A}|U^*)$.

9.1. **Proposition.** If $\mathbf{L}$ is a flabby sheaf and if φ is a paracompactifying family of supports on Y, then $f_\psi \mathbf{L}$ is φ-soft for any family ψ of supports on X.

Proof. Let $s \in (f_\psi \mathbf{L})(K)$, where $K \in \varphi$. By Theorem 4.2, there is an open neighborhood U of K, and an extension $s' \in (f_\psi \mathbf{L})(U) = \Gamma_{\psi(U)}(\mathbf{L}|U^*)$ of s. Since K has a paracompact neighborhood and since paracompact spaces are normal, there is an open neighborhood V of K with $\bar{V} \subset U$ and $\bar{V} \in \varphi$.

Let $s'' = s'|V^* \in \Gamma_{\psi(V)}(\mathbf{L}|V^*)$ and let t be the zero section of $\mathbf{L}$ over $X - (\bar{V}^* \cap |s'|)$. Since s'' and t agree where both are defined and since $\mathbf{L}$ is flabby and φ is paracompactifying, Corollary 4.3 implies that $\mathbf{L}$ is φ-soft. Thus there is an $s^* \in \mathbf{L}(X)$ extending both s'' and t. $|s^*| \in \bar{V}^* \cap |s'| \in \psi(Y)$. Thus we may regard s^* as an element of $(f_\psi \mathbf{L})(Y)$ and s^* has support in $\bar{V} \in \varphi$. Therefore s^* is the required extension of s to $\Gamma_\varphi(f_\psi \mathbf{L})$. Q. E. D.

Let $A \subset X$, i be the inclusion $A \to X$, $\mathbf{A}$ be a sheaf on X, $\mathbf{A}^* = \mathbf{C}^*(X, A; \mathbf{A})$, ψ be a family of supports on X, f: $X \to Y$ be any continuous map. We consider the direct image $f_\psi \mathbf{A}^*$ of $\mathbf{A}^*$ on Y. The derived sheaf $\mathbf{H}^*(f_\psi \mathbf{A}^*)$ on Y is called the Leray sheaf of the map f mod $f|A$. Denote it by $\mathbf{H}^*_\psi(f, f|A; \mathbf{A})$. If A is empty, then denote the Leray sheaf by $\mathbf{H}^*_\psi(f; \mathbf{A})$.

Since $f_\psi \mathbf{A}^*$ is the sheaf generated by the presheaf $U \to \Gamma_{\psi\cap U^*}(\mathbf{A}^*|U^*)$. It follows that $\mathbf{H}^p_\psi(f, f|A; \mathbf{A})$ is the sheaf generated by the presheaf: $U \to H^p(\Gamma_{\psi\cap U^*}(\mathbf{C}^*(X, A; \mathbf{A})|U^*)) = H^p_{U\cap\psi^*}(U^*, U^* \cap A; \mathbf{A}|U^*)$.

For $y \in Y$, let y^* denote $f^{-1}(y)$. Then the restriction map: $H^*_{\psi\cap U}(U^*, U^* \cap A; \mathbf{A}|U^*) \to H^*_{\psi\cap y^*}(y^*, y^* \cap A; \mathbf{A}|y^*)$ induces a homomorphism r^*_y: $\mathbf{H}^*_\psi(f, f|A; \mathbf{A})_y \to H^*_{\psi\cap y^*}(y^*, y^* \cap A; \mathbf{A}|y^*)$.

Since $0 \to \Gamma_{\psi\cap U^*}(\mathbf{C}^*(X, A; \mathbf{A})|U^*) \to \Gamma_{\psi\cap U^*}(\mathbf{C}^*(X; \mathbf{A})|U^*) \to \Gamma_{\psi\cap U^*}(i\,\mathbf{C}^*(A; \mathbf{A}|A)|U^*) \to 0$ is exact. Hence there exists a long exact sequence: $\cdots \to \mathbf{H}^p_\psi(f, f|A; \mathbf{A}) \to \mathbf{H}^p_\psi(f; \mathbf{A}) \to \mathbf{H}^p_{\psi\cap A}(f|A; \mathbf{A}) \to \mathbf{H}^{p+1}_\psi(f, f|A; \mathbf{A}) \to \cdots$ of sheaves on Y. Thus, it suffices to consider only the absolute case in which A is empty.

9.2. **Theorem.** If f is ψ-closed (that is, for any K in ψ, $f(K)$ is closed), and y^* is ψ-taut in X (which we mean that for every flabby sheaf $\mathbf{A}$ on X, the restriction $\Gamma_{\psi}(\mathbf{A})\to\Gamma_{\psi\cap y^*}(\mathbf{A}|y^*)$ is onto and $\mathbf{A}|y^*$ is $(\psi\cap y^*)$-acyclic), then r_y^*: $\mathbf{H}_{\psi}^*(f;\mathbf{A})_y\to\mathrm{H}_{\psi\cap y^*}^*(y^*;\mathbf{A})$ is an isomorphism for all $\mathbf{A}$.

Proof. We consider the functors $\mathrm{F}_1^p(\mathbf{A})=\varinjlim_{U}\mathrm{H}_{\psi\cap U^*}^p(U^*;\mathbf{A}|U^*)$, where U ranges over open sets containing y, and

$$\mathrm{F}_2^p(\mathbf{A})=\mathrm{H}_{\psi\cap y^*}^p(y^*;\mathbf{A}|y^*).$$

For each $K\in\psi$ with $K\cap y^*=\phi$, that is $y\in Y-f(K)$, since f is ψ-closed, $Y-f(K)$ is an open neighborhood of y. Let $U=Y-f(K)$, then $U^*\cap K=\phi$. Therefore, in degree zero, r_y^*: $\mathrm{F}_1^0(\mathbf{A})=\varinjlim\Gamma_{\psi\cap U^*}(\mathbf{A}|U^*)$ $\to\Gamma_{\psi\cap y^*}(\mathbf{A}|y^*)=\mathrm{F}_2^0(\mathbf{A})$ is well-defined and is 1—1 for arbritrary $\mathbf{A}$ and onto for $\mathbf{A}$ flabby (since y^* is ψ-taut). Applying F_1 and F_2 to the sequence: $0\to\mathbf{A}\to\mathbf{C}^0(X;\mathbf{A})\to\mathbf{Z}^1(X;\mathbf{A})\to0$. We obtain a commutative diagram with both rows are exact:

$$\begin{array}{ccccccc}
0\to\mathrm{F}_1^0(\mathbf{A})\to & \mathrm{F}_1^0(\mathbf{C}^0(X;\mathbf{A}))\to & \mathrm{F}_1^0(\mathbf{Z}^1(X;\mathbf{A}))\to & \mathrm{F}_1^1(\mathbf{A}) \\
\downarrow 1-1 & \| & \downarrow 1-1 & \downarrow 1-1 \\
0-\mathrm{F}_2^0(\mathbf{A})- & \mathrm{F}_2^0(\mathbf{C}^0(X;\mathbf{A}))\to & \mathrm{F}_2^0(\mathbf{Z}^1(X;\mathbf{A}))\to & \mathrm{F}_2^1(\mathbf{A})
\end{array}$$

By five lemma[3], $\mathrm{F}_1^0(\mathbf{A})\to\mathrm{F}_2^0(\mathbf{A})$ is onto.
Thus $\mathrm{F}_1^0(\mathbf{A})\to\mathrm{F}_2^0(\mathbf{A})$ is an isomorphism for every sheaf $\mathbf{A}$ on X.

Then applying F_1^* and F_2^* to the sequences: $0\to\mathbf{Z}^p(X;\mathbf{A})\to$ $\mathbf{C}^p(X;\mathbf{A})\to\mathbf{Z}^{p+1}(X;\mathbf{A})\to0$. And since each open set U^* is ψ-taut, we obtain the sequence of homomorphisms: $\mathrm{F}_i^0(\mathbf{C}^{p-1}(X;\mathbf{A}))\to$ $\mathrm{F}_i^0(\mathbf{Z}^p(X;\mathbf{A}))\to\mathrm{F}_i^1(\mathbf{Z}^{p-1}(X;\mathbf{A}))=\mathrm{F}_i^2(\mathbf{Z}^{p-2}(X;\mathbf{A}))=\cdots=\mathrm{F}_i^p(\mathbf{Z}^0(X;\mathbf{A}))=$ $\mathrm{F}_i^p(\mathbf{A})$; (which is exact at the second term).

It yields a surjection $\mathrm{F}_i^0(\mathbf{Z}^p(X;\mathbf{A}))\to\mathrm{F}_i^p(\mathbf{A})$ whose kernel is the image of $\mathrm{F}_i^0(\mathbf{C}^{p-1}(X;\mathbf{A}))$. Thus the isomorphism $\mathrm{F}_1^0(\mathbf{A})\to\mathrm{F}_2^0(\mathbf{A})$ induces an isomorphism $\mathrm{F}_1^p(\mathbf{A})\to\mathrm{F}_2^p(\mathbf{A})$ for each p. Q. E. D.

9.3. **Theorem.** Let $A\subset X$ and f: $X\to Y$ be continuous, and let ψ and φ be families of supports on X and Y respectively, $\mathbf{A}$ be a sheaf on X. If one of the following conditions holds:

(A) ψ consists of all closed sets,

(B) φ is paracompactifying,

then there exists a spectral sequence in which:

$$E_2^{p,q}=H_\varphi^p(Y;\mathbf{H}_\psi^q(f,f|A;\mathbf{A}))\Rightarrow H_{\varphi(\psi)}^{p+q}(X,A;\mathbf{A}),$$

where $\varphi(\psi)=\{K\in\psi(Y):\overline{f(K)}\in\varphi\}=\psi(Y)\cap f^{-1}(\varphi)$.

Proof. Let $\mathbf{A}^*=\mathbf{C}^*(X,A;\mathbf{A})$. Since $\mathbf{A}^*$ consists of flabby sheaves, if in case (A), then by Proposition 3.2 and Theorem 3.4, $f_\psi\mathbf{A}^*=f\mathbf{A}$ consists of φ-acyclic sheaves; if in case (B), then by Proposition 9.1 and Theorem 4.6, $f_\psi\mathbf{A}^*$ also consists of φ-acyclic sheaves. Then let $L^{*,*}$ be the double complex composed of $L^{p,q}=C_\varphi^p(Y;f_\psi\mathbf{A}^q)$. Here we have that $f_\psi\mathbf{A}^q=0$ for $q<0$ and $H^*(H_\varphi^q(Y;f_\psi\mathbf{A}^*))=0$ for $q>0$. Thus by Theorem 8.1, there is a spectral sequence

with $E_2^{p,q}=H_\varphi^p(Y;\mathbf{H}_\psi^q f,f|A;\mathbf{A}))\Rightarrow H^{p+q}(\Gamma_\varphi(f_\psi\mathbf{A}^*))$.

Since $\Gamma_\varphi(f_\psi\mathbf{A}^*)=\Gamma_{\varphi(\psi)}(\mathbf{A}^*)=C_{\varphi(\psi)}^*(X,A;\mathbf{A})$, therefore $E_2^{p,q}\Rightarrow H_{\varphi(\psi)}^{p+q}(X,A;\mathbf{A})$ Q.E.D.

§ 10. LOCALLY TRIVIAL BUNDLES

10.1. **Theorem.** Let f: $X\to Y$ be a locally trivial fiber map with fiber F, ψ be a family of supports on X such that for sufficiently small open set U in Y and any admissible representation of U as a product $U\times F$, $\psi\cap U^*=U\times\theta$, where θ is some fixed family of supports on F, and let $\mathbf{A}$ be a sheaf on X which has the form $U\times\mathbf{B}$ on U for some sheaf $\mathbf{B}$ on F. If each r_y^* (of § 9 with A empty) is an isomorphism. Then the Leray sheaf $\mathbf{H}_\psi^*(f;\mathbf{A})$ is locally constant with stalks $H_\theta^*(F;\mathbf{B})$.

Proof. We may assume that $X=Y\times F$, $\psi=Y\times\theta$ and $\mathbf{A}=Y\times\mathbf{B}$. Let π: $X\to F$ be the canonical projection, so that $\mathbf{A}=\pi^*\mathbf{B}$, where $\pi^*\mathbf{B}$ is the inverse image of $\mathbf{B}$ that is defined by: $\pi^*\mathbf{B}=\{(x,b)\in X\times\mathbf{B}:\pi(x)=\pi_0(b)$, where π_0 is the local homemorphism $\mathbf{B}\to F\}$.

For each open set U in Y, $U^*=U\times F\xrightarrow{\pi}F$, $\mathbf{B}\xrightarrow{\pi^*}\mathbf{A}=\pi^*\mathbf{B}$. Consider each open set set $V\subset F$, we have the induced map π_V^{*0}: $C^0(V;\mathbf{B})\to C^0(\pi^{-1}(V);\mathbf{A})$ defined by mapping the serration s: $V\to\mathbf{B}$ into the serration $\pi_V^{*0}(s)$: $\pi^{-1}(V)\to\mathbf{A}$ given by $\pi_V^{*0}(s)(x)=\pi_x^*(s(\pi(x)))$. These induce a homomorphism $\mathbf{C}^0(F;\mathbf{B})\to\mathbf{C}^0(U^*;\mathbf{A})$ compatible with π^* and the inclusions $\mathbf{A}\to\mathbf{C}^0(U^*;\mathbf{A})$, and $\mathbf{B}\to\mathbf{C}^0(F;\mathbf{B})$. This yields a

homomorphism of the quotient sheaves, and by induction, we obtain homomorphisms: $\mathbf{C}^*(F;\mathbf{B})\to\mathbf{C}^*(U^*;\mathbf{A})$ and induce the chain map $C_\theta(F;\mathbf{B})\to C_{\psi\cap U^*}(U^*;\mathbf{A})$. Hence this give rise to a homomorphism also denoted by π^*: $H^*_\theta(F;\mathbf{B})H^*_{\psi\cap U^*}(U^*;\mathbf{A})$. Thus it again induces a homomorphism from the constant sheaf $H^*_\theta(F;\mathbf{B})$ to $\mathbf{H}^*_\psi(f;\mathbf{A})$. On the stalk at y, this homomorphism π^*_y: $H^*_\theta(F;\mathbf{B})\to\varinjlim_{y\in U} H^*_{\psi\cap U^*}(U^*;\mathbf{A})=\mathbf{H}^*_\psi(f;\mathbf{A})_y$ is the inverse of r^*_y: $\mathbf{H}^*_\psi(f;\mathbf{A})_y\to H^*_{\psi\cap y^*}(y^*;\mathbf{A})=H^*_\theta(F;\mathbf{B})$. $(y^*=f^{-1}(y)=F)$. Thus $\mathbf{H}^*_\psi(f;\mathbf{A})$ is locally constant with stalks $H^*_\theta(F;\mathbf{B})$. Q. E. D.

10.2. **Theorem.** If π: $X\to Y$ is an orientable n-sphere bundle mapping[4] $(n\geq 1)$. Let ψ be a family of supports on X consisting of all closed sets, and φ be any family of supports on Y, $\mathbf{B}$ be any sheaf on Y. Then we have the Gysin sequence:

$$\cdots\to H^{k-1}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})\to H^{k-n-1}_\varphi(Y;\mathbf{B})\to H^k_\varphi(Y;\mathbf{B})\to H^k_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})$$
$$\to H^{k-n}_\varphi(Y;\mathbf{B})\to H^{k+1}_\varphi(Y;\mathbf{B})\to H^{k+1}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})\to\cdots$$

Proof. By Theorem 9.3, there exists a spectral sequence in which: $E_2^{p,q}=H^p_\varphi(Y;\mathbf{H}^q_\psi(\pi;\pi^*\mathbf{B}))\Rightarrow H^{p+q}_{\varphi(\psi)}(X;\pi^*\mathbf{B})=H^{p+q}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})$.

Let $y\in Y$, for each $K\in\psi$ with $K\cap\pi^{-1}(y)=\phi$. Then by the locally trivial property and the compactness of $S^n\cong\pi^{-1}(y)$, there exists a small open set $U\subset Y$ with $y\in U$ and $K\cap\pi^{-1}(U)=\phi$. By Corollary 4.8 and since $\psi\cap S^n$ is paracompactifying, $S^n=f^{-1}(y)$ is ψ-taut. Thus from Theorem 9.2, we know that r^*_y: $\mathbf{H}^*_\psi(\pi,\pi^*\mathbf{B})_y\to H^*_{\psi\cap y^*}(y^*;\pi^*\mathbf{B})$ is an isomorphism. By Theorem 10.1, $\mathbf{H}^*_\psi(\pi,\pi^*\mathbf{B})$ is locally constant with stalks $H^*_\theta(S^n;\mathbf{B})$.

Therefore, $E_2^{p,q}=0$ for $q\neq 0$ $q\neq n$, and $E_2^{p,0}=E_2^{p,n}\cong H^p_\varphi(Y;\mathbf{B})$. Hence $E_\infty^{p,q}=F_pH^{p+q}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})/F_{p+1}H^{p+q}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})=0$ for $q\neq 0$, $q\neq n$.
Set $p+q=k$, then

$$E_\infty^{k-q,q}=F_{k-q}H^k_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})/F_{k-q+1}H^k_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})$$
$$=F_{k-q}H^k/F_{k-q+1}H^k,\quad\text{and}$$
$$0=\cdots=F_{k+1}H^k\subset F_kH^k=F_{k-1}H^k=\cdots$$
$$=F_{k-n+1}H^k\subset F_{k-n}H^k=\cdots=H^k_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})=H^k.$$

Then $E_\infty^{k,0}=F_kH^k$ and
$E_\infty^{k-n,n}=H^k/F_kH^k$.

And thus (A): $E_\infty^{k,0}\to H^k\to E_\infty^{k-n,n}\to 0$ is exact. Consider the two diagrams:

$$0=\left\{\begin{array}{ccccc} E_2^{k-2,1} & \longrightarrow & E_2^{k,0} & \to & E_2^{k+2,-1} \\ & & \| & & \\ & & E_3^{k,0} & & \\ & & \| & & \\ & & \vdots & & \\ & & \| & & \\ E_n^{k-n,n-1} & \to & E_n^{k,0} & \to & E_n^{k+n,-n+1} \end{array}\right\}=0$$

$$0\neq E_{n+1}^{k-n-1,n}\to E_{n+1}^{k,0}$$

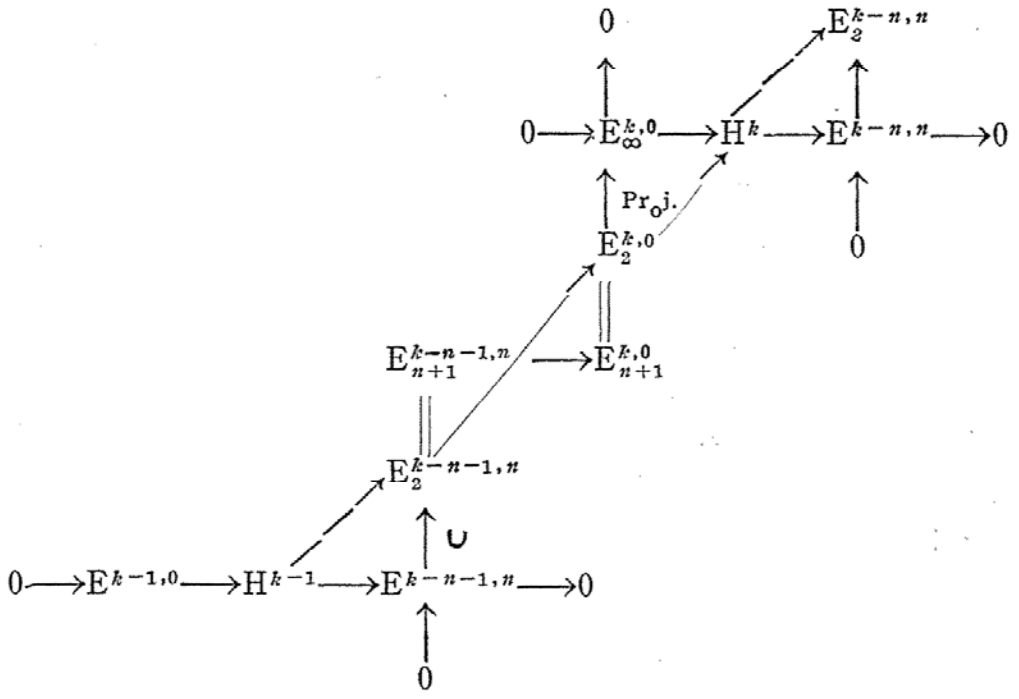
$$\downarrow \text{Proj.}$$

$$0=\left\{\begin{array}{c} E_{n+2}^{k,0} \\ \| \\ \vdots \\ \| \\ E_\infty^{k,0} \end{array}\right\}$$

$$0=\left\{\begin{array}{ccc} \to E_2^{k-n,n} & \to & E_2^{k-n+2,n-1} \\ \| & & \\ E_3^{k-n,n} & & \\ \| & & \\ \vdots & & \\ \| & & \\ E_n^{k-n,n} & \to & E_n^{k,1} \end{array}\right\}=0$$

$$0=\left\{\begin{array}{c} E_{n+1}^{k-n,n}\to E_{n+1}^{k+1,0}\neq 0 \\ \uparrow \cup \\ E_{n+2}^{k-n,n} \\ \| \\ \vdots \\ \| \\ E_\infty^{k-n,n} \end{array}\right\}=0$$

Thus we can form the following diagram by the above two diagrams and the sequence (A);



And obtain the exact sequence:

$$\cdots\to H^{k-1}\to E_2^{k-n-1,n}\to E_2^{k,0}\to H^k\to E_2^{k-n,n}\to E_2^{k+1,0}\to H^{k+1}\to\cdots$$

That is the Gysin exact sequence:

$$\cdots\to H^{k-1}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})\to H^{k-n-1}_{\varphi}(Y;\mathbf{B})\to H^{k}_{\varphi}(Y;\mathbf{B})\to H^{k}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})\to$$
$$H^{k-n}_{\varphi}(Y;\mathbf{B})\to H^{k+1}_{\varphi}(Y;\mathbf{B})\to H^{k+1}_{\pi^{-1}(\varphi)}(X;\pi^*\mathbf{B})\to\cdots$$

Q. E. D.

10.3. **Theorem.** Let π: $X\to S^n(n\geq 2)$ be a bundle map with fiber F, L be a fixed ground ring which is a principal ideal domain. And let ψ, θ be families on X, F respectively as in Theorem 10.1. Then we have the Wang exact sequence:

$$\cdots\to H^{k}_{\psi}(X)\to H^{k}_{\theta}(F)\to H^{k-n-1}_{\theta}(F)\to H^{k+1}_{\psi}(X)\to\cdots$$

Proof. Since there exists a spectral sequence:

$$E_2^{p,q}=H^p(S^n;\mathbf{H}^q_{\psi}(\pi;L))\Rightarrow H^{p+q}_{\psi}(X;L).$$

But $\mathbf{H}^*_{\psi}(\pi;L)$ is locally constant with stalks $H^*_{\theta}(F;L)$.

Thus $E_2^{p,q}=0$ for $p\neq 0$, $p\neq n$, and $E_2^{0,q}=E_2^{n,q}=H^q_{\theta}(F;L)$. The remainder of the proof is the same as in Theorem 10.2. Q. E. D.

REFERENCES

(1) D. G. Bourgin, *Modern Algebraic Topology*, Macmillan (1963).
(2) H. Cartan, and S. Eilenberg, *Homological Algebra*, Princeton Press (1956).
(3) E. H. Spainer, *Algebraic Topology*, McGraw-Hill (1966).
(4) N. Steenrod, *The Topology of Fibre Bundles*, Princeton Univ. Press (1951).

"Beauty is truth, truth beauty, — that is all Ye know on Earth, and all Ye need to know."

John Keats

THE $\pi\pi$ SIGMA TERM AND SCATTERING LENGTHS*

I-FU SHIH

ABSTRACT

S. Weinberg's current algebra calculation of the $\pi\pi$ scattering lengths involves an assumption of an *isoscalar* σ field and utilization of Adler's self-consistency condition. These are not well justified. We show here that the new technique of the dispersion inequality can be applied to bound the $\pi\pi$ sigma term. This offers a check on Weinberg's calculation. Also this enables us to bound the scattering lengths without the *isoscalar* assumption of the σ field. The results are $|a_0^{(0)}| \lesssim 0.35 m_\pi^{-1}$ and $|a_0^{(2)}| \lesssim 0.03 m_\pi^{-1}$.

Finally, we propose a way to further avoid utilizing Adler's self-consistency condition.

1. INTRODUCTION

Several years ago S. Weinberg[1] calculated the pion-pion scattering lengths by means of current algebra and PCAC.[2] The results are in general consistent with most of the recent experimental data. Nevertheless, two assumptions which are not well justified were made in this calculation. One is the assumption that the $\pi\pi$ σ-term is an isoscalar and the other is the assumed truth of Adler's self-consistency condition.[3]

Here we shall show that a recently developed technique, the dispersion inequality,[4] allows us to estimate the magnitude of the $\pi\pi$ σ-term. This serves as an examination of the assumption mentioned above, and also enables us to estimate the $\pi\pi$ scattering lengths without the isoscalar assumption and Adler's condition.

We shall first review Weinberg's calculation and, in the mean time, supply many of missing steps in his highly condensed paper.

2. CURRENT ALGEBRA CALCULATION OF THE PI-PI SCATTERING AMPLITUDE

Let us consider the following scattering process

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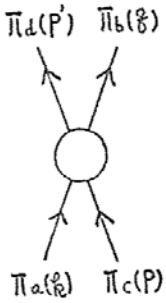

Fig. 1. The $\pi\pi$ Scattering Process

$$\pi_a(k)+\pi_c(p)\longrightarrow\pi_b(q)+\pi_d(p'), \tag{1}$$

where a, b, c, and d are pion isovector indices (running over 1, 2, 3) and k, q, p, and p' are the momenta of corresponding pions. We write the S-matrix in terms of the invariant amplitude $\langle\pi_d(p')\pi_b(q)|\mathrm{M}|\pi_c(p)\pi_a(k)\rangle$,

$$\begin{aligned}&(16q^0k^0p^0p'^0\mathrm{V}^4)^{1/2}\langle\pi_d(p')\pi_b(q)|\mathrm{S}|\pi_c(p)\pi_a(k)\rangle\\&\quad=-i(2\pi)^4\delta^4(p+k-p'-q)\langle\pi_d(p')\pi_b(q)|\mathrm{M}|\pi_c(p)\pi_a(k)\rangle.\end{aligned} \tag{2}$$

Because we shall later consider some pions to be soft, the amplitude to be dealt with is actually not on the physical threshold. It is thus convenient to define, in accordance with LSZ reduction formalism[5] and PCAC hypothesis, the off-mass-shell invariant amplitude by

$$\begin{aligned}&\int d^4xd^4ye^{-iqx}e^{iky}\langle\pi_d(p')|\mathrm{T}\{\partial_\mu\mathrm{A}_b^\mu(x),\ \partial_\nu\mathrm{A}_a^\nu(y)\}|\pi_c(p)\rangle\\&\quad\equiv\frac{i(2\pi)^4\delta^4(p+k-p'-q)\mathrm{F}_\pi^2m_\pi^4}{(q^2+m_\pi^2)(k^2+m_\pi^2)(4p^0p'^0\mathrm{V}^2)^{1/2}}\\&\quad\langle\pi_d(p')\pi_b(q)|\mathrm{M}|\pi_c(p)\pi_a(k)\rangle,\end{aligned} \tag{3}$$

where F_π is the pion decay constant, defined by

$$(2q^0\mathrm{V})^{1/2}\langle0|\partial_\nu\mathrm{A}_a^\nu(0)|\pi_b(q)\rangle\equiv\mathrm{F}_\pi m_\pi^2\delta_{ab}. \tag{4}$$

Experiment gives $\mathrm{F}_\pi\approx94$ Mev or $0.67\ m_\pi$.

Using the current commutation rules

$$\delta(x^0-y^0)[\mathrm{A}_a^0(y),\ \mathrm{A}_b^\mu(x)]=i\varepsilon_{abc}\mathrm{V}_c^\mu(x)\delta^4(x-y), \tag{5}$$

$$\delta(x^0-y^0)[\mathrm{A}_b^0(x),\ \partial_\nu\mathrm{A}_a^\nu(y)]=i\sigma_{ab}(x)\delta^4(x-y), \tag{6}$$

where $\sigma_{ab}(x)$ can be shown to be symmetric, i.e., $\sigma_{ab}(x)=\sigma_{ba}(x)$, and also noting the identity

$$\frac{\partial}{\partial x^\lambda}T\{A^\lambda(x),\ B(y)\}\equiv T\{\partial_\lambda A^\lambda(x),\ B(y)\}+\delta(x^0-y^0)[A^0(x),\ B(y)],\qquad(7)$$

we see that the time-ordered product appearing in Eq. (3) can be expressed as

$$\begin{aligned}
&T\{\partial_\mu A^\mu_b(x),\ \partial_\nu A^\nu_a(y)\}\\
&=\frac{1}{2}\left(\frac{\partial}{\partial x^\mu}\ \frac{\partial}{\partial y^\nu}+\frac{\partial}{\partial y^\nu}\ \frac{\partial}{\partial x^\mu}\right)T\{A^\mu_b(x),\ A^\nu_a(y)\}\\
&\quad-\frac{1}{2}\{\delta(x^0-y^0)[A^0_b(x),\ \partial_\nu A^\nu_a(y)]\\
&\quad+\delta(x^0-y^0)[A^0_a(y),\ \partial_\mu A^\mu_b(x)]\}\\
&\quad-\frac{1}{2}\left\{\frac{\partial}{\partial y^\nu}\delta(x^0-y^0)[A^0_b(x),\ A^\nu_a(y)]\right.\\
&\quad\left.+\frac{\partial}{\partial x^\mu}\delta(x^0-y^0)[A^0_a(y),\ A^\mu_b(x)]\right\}\\
&=\frac{1}{2}\left(\frac{\partial}{\partial x^\mu}\ \frac{\partial}{\partial y^\nu}+\frac{\partial}{\partial y^\nu}\ \frac{\partial}{\partial x^\mu}\right)T\{A^\mu_b(x),\ A^\nu_a(y)\}\\
&\quad-i\sigma_{ab}(x)\delta^4(x-y)\\
&\quad-\frac{1}{2}\left(\frac{\partial}{\partial x^\mu}-\frac{\partial}{\partial y^\mu}\right)i\varepsilon_{abm}V^\mu_m(x)\delta^4(x-y).
\end{aligned}\qquad(8)$$

Substituting this into the left-hand side of Eq. (3) and integrating by parts gives

$$\begin{aligned}
&\int d^4x d^4y e^{-iq_x}e^{ik_y}\langle\pi_d(p')|T\{\partial_\mu A^\mu_b(x),\ \partial_\nu A^\nu_a(y)\}|\pi_c(p)\rangle\\
&=q_\mu k_\nu\int d^4x d^4y e^{-iq_x}e^{ik_y}\langle\pi_d(p')|T\{A^\mu_b(x),\ A^\nu_a(y)\}|\pi_c(p)\rangle\\
&\quad-i\int d^4x d^4y e^{-iq_x}e^{ik_y}\delta^4(x-y)\langle\pi_d(p')|\sigma_{ab}(x)|\pi_c(p)\rangle\\
&\quad+\varepsilon_{abm}\frac{(q+k)_\mu}{2}\int d^4x d^4y e^{-iq_x}e^{ik_y}\delta^4(x-y)\\
&\quad\langle\pi_d(p')|V^\mu_m(x)|\pi_c(p)\rangle\\
&=i(2\pi)^4\delta^4(p+k-p'-q)\{-iq_\mu k_\nu\int d^4x' e^{-iq_{x'}}\\
&\quad\langle\pi_d(p')|T\{A^\mu_b(x')A^\nu_a(0)\}|\pi_c(p)\rangle\\
&\quad-\langle\pi_d(p')|\sigma_{ab}(0)|\pi_c(p)\rangle\\
&\quad-i\varepsilon_{abm}\frac{(q+k)_\mu}{2}\langle\pi_d(p')|V^\mu_m(0)|\pi_c(p)\rangle\}.
\end{aligned}\qquad(9)$$

When we keep $\pi_c(p)$ and $\pi_d(p')$ on the mass shell and let $\pi_a(k)$ and $\pi_b(q)$ be soft, i.e., $q^\mu=k^\mu\to 0$ (so $p^\mu=p'^\mu$), the first term in Eq. (9) vanishes because there is no pole term in this case. The third term becomes

$$\begin{aligned}&-i\varepsilon_{abm}q_\mu\langle\pi_d(p')|V_m^\mu(0)|\pi_c(p)\rangle|_{p=p'}\\&\quad=-i\varepsilon_{abm}q_\mu\cdot i\varepsilon_{dmc}(2p_\mu)(4p^0p'^0V^2)^{-1/2}\\&\quad=-2(p\cdot q)(\delta_{ad}\delta_{bc}-\delta_{ac}\delta_{bd})(4p^0p'^0V^2)^{-1/2}.\end{aligned}\tag{10}$$

Eq. (3) shows that the invariant amplitude approaches

$$\begin{aligned}&\langle\pi_d(p')\pi_b(q)|M|\pi_c(p)\pi_a(k)\rangle\longrightarrow\\&M^{(0)}_{db,ca}-\frac{2}{F_\pi^2}(p\cdot q)(\delta_{ad}\delta_{bc}-\delta_{ac}\hat{\delta}_{bd}),\end{aligned}\tag{11}$$

where

$$M^{(0)}_{db,ca}=-\frac{1}{F_\pi^2}(4p^0p'^0V^2)^{1/2}\langle\pi_d(p')|\sigma_{ab}(0)|\pi_c(p)\rangle|_{p=p'}.\tag{12}$$

3. SCATTERING LENGTHS

From the consideration of crossing symmetry, isospin conservation, and Bose statistics, the expansion of the invariant amplitude to second order in momenta can be written in the form:

$$\begin{aligned}&\langle\pi_d(p')\pi_b(q)|M|\pi_c(p)\pi_a(k)\rangle\\&\quad=\delta_{ab}\delta_{cd}[A+B(s+u)+Ct]+\delta_{ad}\delta_{cb}[A+B(s+t)+Cu]\\&\qquad+\delta_{ac}\delta_{bd}[A+B(u+t)+Cs]+\text{“higher order”},\end{aligned}\tag{13}$$

where A, B, and C are constant coefficients and

$$s\equiv-(p+k)^2,\qquad t\equiv-(k-q)^2,\qquad u\equiv-(p-q)^2.\tag{14}$$

Since isospin is conserved, the amplitude can be divided into three parts corresponding to three possible isospin states T=0, 1, and 2,

$$\langle\pi_d(p')\pi_b(q)|M|\pi_c(p)\pi_a(k)\rangle=\sum_{T=0}^{2}F^{(T)}P^{(T)}_{db,ca}.\tag{15}$$

The isospin projection operators[6] are

$$P^{(0)}_{db,ca}=\frac{1}{3}\delta_{ac}\delta_{bd},$$
$$P^{(1)}_{db,ca}=\frac{1}{2}\delta_{ab}\delta_{cd}-\frac{1}{2}\delta_{ad}\delta_{cb}, \tag{16}$$
$$P^{(2)}_{db,ca}=\frac{1}{2}\delta_{ab}\delta_{cd}+\frac{1}{2}\delta_{ad}\delta_{cb}-\frac{1}{3}\delta_{ac}\delta_{bd}.$$

After a simple calculation we will find that

$$F^{(0)}=5A+(2s+4u+4t)B+(3s+u+t)C,$$
$$F^{(1)}=(u-t)B+(t-u)C, \tag{17}$$
$$F^{(2)}=2A+(2s+u+t)B+(u+t)C.$$

It is most convenient to express these three amplitudes $F^{(T)}$ (T=0, 1, and 2) corresponding to well-defined isospins in the form of a partial wave expansion[7],

$$F^{(T)}(s,\cos\theta)=\sum_l(2l+1)F^{(T)}_l(s)P_l(\cos\theta). \tag{18}$$

Unitarity allows us to write $F^{(T)}_l(s)$ in terms of phase shifts

$$F^{(T)}_l(s)=-\frac{16\pi\sqrt{s}}{q}e^{i\delta^{(T)}_l(s)}\sin\delta^{(T)}_l(s). \tag{19}$$

The scattering length is, by definition,

$$a^{(T)}_l=\lim_{q\to 0}\frac{e^{i\delta^{(T)}_l(s)}\sin\delta^{(T)}_l(s)}{q^{2l+1}} \tag{20}$$

At the low energy limit only the s-wave is important. Also, on the physical threshold $s=4m_\pi^2$, $t=u=0$, so

$$F^{(0)}\approx F^{(0)}_0(s)P_0(\cos\theta)=-\frac{16\pi\sqrt{s}}{q}e^{i\delta^{(0)}_0(s)}\sin\delta^{(0)}_0(s). \tag{21}$$

On the other hand, Eq. (17) gives

$$F^{(0)}=5A+8m_\pi^2B+12m_\pi^2C, \tag{22}$$

therefore we have

$$a^{(0)}_0\simeq-\frac{1}{32\pi m_\pi}[5A+8m_\pi^2B+12m_\pi^2C]. \tag{23}$$

Similarly, we also find $a^{(1)}_0=0$ and

$$a_0^{(2)} \simeq -\frac{1}{32\pi m_\pi}[2A+8m_\pi^2 B]. \tag{24}$$

In order to compare Eq. (13) with the current algebra calculation, we let $\pi_c(p)$ and $\pi_d(p')$ be on the mass shell and consider $\pi_a(k)$ and $\pi_b(q)$ to be soft, i.e., $q_\mu = k_\mu \to 0$ and $p_\mu = p'_\mu$ or equivalently

$$t \to 0; \qquad s \to m_\pi^2 - 2p\cdot k = m_\pi^2 - 2p\cdot q; \qquad u \to m_\pi^2 + 2p\cdot q. \tag{25}$$

Now, the invariant amplitude approaches

$$\begin{aligned}\langle \pi_d(p')\pi_b(q)|M|\pi_c(p)\pi_a(k)\rangle &= \delta_{ab}\delta_{cd}[A+2m_\pi^2 B]+(\delta_{ad}\delta_{cb}+\delta_{ac}\delta_{bd})[A+m_\pi^2 C+m_\pi^2 B] \\ &\quad -2(p\cdot q)(B-C)[\delta_{ad}\delta_{cb}-\delta_{ac}\delta_{bd}].\end{aligned} \tag{26}$$

Eq. (11) and Eq. (26) lead to

$$B-C=\frac{1}{F_\pi^2} \tag{27}$$

and

$$\begin{aligned}M^{(0)}_{bd,ca} &= \delta_{ab}\delta_{cd}[A+2m_\pi^2 B] \\ &\quad +(\delta_{ad}\delta_{bc}+\delta_{ac}\delta_{bd})[A+m_\pi^2 C+m_\pi^2 B].\end{aligned} \tag{28}$$

To determine the scattering lengths, or the coefficients A, B, and C, Weinberg assumed that the sigma field $\sigma_{ab}(x)$ is an isoscalar, i.e., $\sigma_{ab}(x)=\delta_{ab}\sigma(x)$, so that $M^{(0)}_{db,ca}$ is only proportional to δ_{ab}. Eq. (28) then gives

$$A=-m_\pi^2(B+C). \tag{29}$$

Also used is the Adler's self-consistency condition which states that the amplitude vanishes when any one of the four pion momenta vanishes and the other three are on the mass shell, i.e., $\langle \pi_d(p')\pi_b(q)|M|\pi_c(p)\pi_a(k)\rangle = 0$, when $s=t=u=m_\pi^2$. So, from Eq. (13),

$$A=-m_\pi^2(2B+C). \tag{30}$$

Eqs. (21), (29), and (30) yield

$$A=\frac{m_\pi^2}{F_\pi^2}; \qquad B=0; \qquad C=-\frac{1}{F_\pi^2}, \tag{31}$$

and then

$$a_0^{(0)} \simeq \frac{7m_\pi^2}{32\pi m_\pi F_\pi^2} = 0.16\ m_\pi^{-1},$$
$$a_0^{(2)} \simeq -\frac{2m_\pi^2}{32\pi m_\pi F_\pi^2} = -0.05 m_\pi^{-1}. \tag{32}$$

Here we have used the experimental value $F_\pi \simeq 0.67\ m_\pi$ rather than the theoretical value from the Goldberger-Treiman relation. These results are in general consistent with experimental data.

4. ESTIMATE OF THE PI-PI SIGMA TERM

Under the isoscalar assumption for $\sigma_{ab}(x)$, we may define a form factor $F(t)$ for the $\pi\pi$ σ-term by

$$(4p^0p'^0V^2)^{1/2}\langle \pi_d(p')|\sigma_{ab}(0)|\pi_c(p)\rangle = -\delta_{ab}\delta_{cd}F(t);$$
$$t = -(p-p')^2. \tag{33}$$

Then Eq. (12) simply states that

$$M^{(0)}_{db,ca} = \frac{1}{F_\pi^2}F(0)\delta_{ab}\delta_{cd}. \tag{34}$$

On the other hand, Eqs. (31) and (28) give us

$$M^{(0)}_{db,ca} = \frac{m_\pi^2}{F_\pi^2}\delta_{ab}\delta_{cd}. \tag{35}$$

So we have

$$F(0) = m_\pi^2. \tag{36}$$

We would like to remark here that Eq. (36) is simply a direct consequence of the isoscalar assumption and has nothing to do with Adler's self-consistency condition. It can be seen that Eqs. (29) and (27) are sufficient to yield Eq. (36).

This result is consistent with the (3, 3*)+(3*, 3) model.[8] In this model, the symmetry breaking part of the Hamiltonian is $\mathcal{H}' = -u_0 - cu_8$. Consequently we can show that

$$\sigma_{ab}(0) = -\delta_{ab}\left(\frac{\sqrt{2}+c}{3}\right)(u_8 + \sqrt{2}u_0). \tag{37}$$

Using Gell-Mann, Oakes, and Renner's formula

$$-(4p^0p'^0V)^{1/2}\langle\pi_i(p)|u_j|\pi_k(p)\rangle=\alpha\delta_{j0}\delta_{ik}+\beta d_{ijk};$$
$$\alpha\approx 0, \qquad \beta\approx\Delta m^2/c, \qquad c=-1.25, \tag{38}$$

we will get $F(0)\cong m_\pi^2$.

Since the technique of dispersion inequality[4] allows us to find a bound on the form factor $F(0)$, or a bound on the $\pi\pi$ σ-term from the propagator of $\sigma(x)$, it is therefore interesting to investigate whether the bound obtained from this new technique and the above results are consistent.

Let us start with the propagator of $\sigma(x)$ and assume that it satisfies an unsubtract dispersion relation,

$$\Delta(q^2)=i\int d^4x e^{iq\cdot x}\langle 0|T\{\sigma(x)\sigma(0)\}|0\rangle$$
$$=\int_{t_0}^{\infty}dt\frac{\rho(t)}{t+q^2}; \quad t_0=4m_\pi^2, \tag{39}$$

where

$$\rho(t)=(2\pi)^3\sum_n\delta^4(p_n-q)|\langle 0|\sigma(0)|n\rangle|^2. \tag{40}$$

If we take only the two-pion intermediate states, then

$$\rho(t)\geqslant\rho^{\pi\pi}(t)=\frac{3}{32\pi^2}\left(\frac{t-t_0}{t}\right)^{1/2}|F(t)|^2; \qquad t\geqslant t_0, \tag{41}$$

where $F(t)$ is the form factor defined in Eq. (33). From Eq. (41) we can form a standard inequality,

$$\frac{1}{\pi}\int_{t_0}^{\infty}dtk(t)|F(t)|^2\leqslant\Delta(0), \tag{42}$$

with

$$k(t)=\frac{3}{32\pi}t^{-3/2}(t-t_0)^{1/2}. \tag{43}$$

The formulas provided in the appendix of reference (4) immediately give us an upper bound of the absolute value of $F(t)$ in the region $t\leqslant t_0$,

$$|F(t)|\leqslant 2\left[\frac{2\pi}{3}\Delta(0)\right]^{1/2}(1+\beta)^{3/2}; \qquad \beta=\left(\frac{t_0}{t_0-t}\right)^{1/2}. \tag{44}$$

We need only a particular case here, namely at $t=0$. Putting $t=0$ in Eq. (44), we have

$$|F(0)|\leqslant 8\left(\frac{\pi}{3}\right)^{1/2}\Delta^{1/2}(0). \tag{45}$$

Now we have to estimate $\Delta^{1/2}(0)$ in order to get the bound of $|F(0)|$. As far as this is concerned, we would first assume a very simple model, where the spectral function is dominated by a zero-width σ resonance which is an isoscalar, scalar meson with a mass of about 700 Mev. In this model then

$$\rho(t)\approx\delta(m_\sigma^2-t)g_\sigma^2, \tag{46}$$

where

$$g_\sigma\equiv(2p^0V)^{1/2}\langle 0|\sigma(0)|\sigma(p)\rangle. \tag{47}$$

We adopt the value of g_σ as that calculated by Amatya, et. al.[9]

$$g_\sigma=m_\pi^4F_\pi^2=0.46\ m_\pi^6. \tag{48}$$

Then we get

$$\Delta(0)=\frac{g_\sigma^2}{m_\sigma^2}=0.017\ m_\pi^4. \tag{49}$$

Eq. (45) therefore leads to

$$|F(0)|\leqslant 1.06\ m_\pi^2 \tag{50}$$

which is in agreement with Weinberg's result.

The zero-width σ-dominance may not be realistic, hence we would like to try a more realistic model. First, let us note that Eq. (41) may be developed to an inequality

$$\frac{1}{\pi}\int_{t_0}^{\infty}dtk(t)|F(t)|^2\leqslant I^2, \tag{51}$$

where

$$I^2=\frac{32\pi}{3}\int_{t_0}^{\infty}dt\left(\frac{t}{t-t_0}\right)^{1/2}\rho(t)k(t) \tag{52}$$

and $k(t)$ is an arbitrary positive weight function. With this kind of dispersion inequality we can take into account the width of the

σ resonance[10]. The bound of $|F(0)|$ derived from this inequality is simply[10]

$$|F(0)|^2 \leqslant \frac{1}{4t_0} I^2 \exp\left\{\frac{t_0}{\pi}^{1/2} \int_{t_0}^{\infty} dt \frac{\ln k(t)}{t(t-t_0)^{1/2}}\right\}. \tag{53}$$

Suppose $\rho(t)$ is dominated by a σ resonance with the width Γ, we can then write $\rho(t)$ in the Breit-Wigner form,

$$\rho(t) = \frac{\frac{m_\sigma \Gamma}{2\pi} g_\sigma^2}{(t-m_\sigma^2)^2 + \frac{m_\sigma^2 \Gamma^2}{4}}. \tag{54}$$

This is normalized to $\int_{-\infty}^{\infty} \rho(t)dt = g_\sigma^2$ in accordance with the zero-width case. If we take

$$k(t) = \frac{3}{32\pi}\left(\frac{t-t_0}{t}\right)^{1/2}\left[(t-m_\sigma^2)^2 + \frac{m_\sigma^2\Gamma^2}{4}\right]^{-1}\frac{1}{t} \tag{55}$$

which is the most reasonable and the best weight function, as has been shown in reference (10), and put $m_\sigma = 700$ Mev, $\Gamma = 300$ Mev, we shall get

$$|F(0)| \leqslant 3.86\ m_\pi^2. \tag{56}$$

This is, of course, not contrary to the former results.

5. THE SCATTERING LENGTHS WITHOUT ISOSCALAR ASSUMPTION

There is no a priori reason for the σ-term to be an isoscalar. Assuming that the σ-term may have both isoscalar and isotensor parts, we write

$$(4p^0 p'^0 V^2)^{1/2} \langle \pi_d(p') | \sigma_{ab}(0) | \pi_c(p) \rangle$$
$$= -\frac{1}{3}\delta_{ab}\delta_{cd} F_0(t) - \frac{1}{2}(\delta_{ad}\delta_{bc} + \delta_{bd}\delta_{ac} - \frac{2}{3}\delta_{ab}\delta_{cd}) F_2(t). \tag{57}$$

In this case, from Eqs. (12), (57), and (28), we have

$$\frac{1}{F_\pi^2} F_0(0) = 5A + 8m_\pi^2 B + 2m_\pi^2 C, \tag{58}$$

$$\frac{1}{F_\pi^2}F_2(0)=2[A+m_\pi^2(B+C)]. \tag{59}$$

Note that in case $F_2(0)\equiv 0$, i. e. only the isoscalar part, $F(0)=\frac{1}{3}F_0(0)$.

Now, we solve for A, B, and C from Eqs. (27), (58), and (30). The result is

$$\begin{aligned} A&=-\frac{4}{5}\frac{m_\pi^2}{F_\pi^2}+\frac{3}{5}\frac{F_0(0)}{F_\pi^2},\\ B&=\frac{3}{5}\frac{1}{F_\pi^2}-\frac{1}{5m_\pi^2}\frac{F_0(0)}{F_\pi^2},\\ C&=-\frac{2}{5}\frac{1}{F_\pi^2}-\frac{1}{5m_\pi^2}\frac{F_0(0)}{F_\pi^2}. \end{aligned} \tag{60}$$

Substituting Eq. (60) into Eqs. (23) and (24) we obtain

$$\begin{aligned} a_0^{(0)}&\simeq\frac{m_\pi^2}{32\pi m_\pi F_\pi^2}\left[4+\frac{F_0(0)}{m_\pi^2}\right],\\ a_0^{(2)}&\simeq\frac{m_\pi^2}{32\pi m_\pi F_\pi^2}\left[\frac{16}{5}-\frac{2}{5}\frac{F_0(0)}{m_\pi^2}\right]. \end{aligned} \tag{61}$$

Since $F_0(0)$ is the isoscalar part of the σ-term, we can now use the bound of $F(0)$; but we must remember that $F(0)=\frac{1}{3}F_0(0)$. If we use the bound obtained in Eq. (56) we will have

$$\begin{aligned} |a_0^{(0)}|&\lesssim 0.35\ m_\pi^{-}\\ |a_0^{(2)}|&\lesssim 0.03\ m_\pi^{-1} \end{aligned} \tag{62}$$

Finally, we simply remark that the use of Adler's self-consistency condition, Eq. (30), may not be justified, since this requires three of the four pions to be on the mass shell, while in the current algebra calculation we consider two pions to be soft. If somehow we can estimate the isotensor part of the σ-term, i. e. $F_2(0)$, then we can solve for A, B, and C from Eqs. (27), (58), and (59), and find a bound on the scattering lengths. This avoids using the Adler self-consistency condition.

ACKNOWLEDGEMENT

I would like to thank Prof. S. Okubo for suggesting this work.

REFERENCES

(1) S. Weinberg, *Phys. Rev. Lett.* **17**, 616 (1966).
(2) See, e. g., S. Adler and R. Dashen, *Current Algebra*, Benjamin Inc. New York (1968).
(3) S. L. Adler, *Phys. Rev.* **137**, B1022 (1965); **139**, B1638 (1965).
(4) S. Okubo and I. F. Shih, *Phys. Rev.* **D4**, 2020 (1971).
(5) H. Lehmann, K. Symanzik, and W. Zimmerman, *Nuovo Cimento* **1**, 205 (1955).
(6) See, e. g., S. Gasiorowicz, *Elementary Particle Physics*, Wiley, N. Y. (1968).
(7) See, e. g., G. Barton, *Dispersion Techniques in Field Theory*, Benjamin New York, (1965).
(8) Gell-Mann, Oakes, and Renner, *Phys. Rev.* **175**, 2195 (1968).
(9) Amatya, Pagnamenta, and Renner, *Phys. Rev.* **172**, 1755 (1968).
(10) See the argument given in I. F. Shih and S. Okubo, *Phys. Rev.* **D6**, 1393 (1972).

THE MASS SPECTRUM OF BARYONS

JEN-I CHEN

ABSTRACT

A new phenomenological approach has been introduced in the study of mass spectra of hadrons. A universal mass formula for all baryons and resonance states is thus obtained.

The study of strong interaction so far still remains in the stage of phenomenological analysis of mass spectra of hadrons and their scattering amplitudes. Both are closely related, in a way, and may be considered as the diagonal and the off-diagonal elements of the same matrix of the still unknown Hamiltonian of strong interaction. The knowledge of mass spectra definitely will reveal important information about strong interaction. Furthermore, the analysis of mass spectra is much more simple, and more basic in view of the fact that through certain reasonable assumptions, such as dispersion relations or thermodynamic models of hadrons which we are attempting, the scattering amplitudes can be calculated if mass spectra are known.

Among enormous particles discovered in recent years, certain empirical regularities of the quantum numbers and masses of these particle states have been suggested. The most successful ones are the SU(3) and the Regge schemes. The SU(3) scheme groups particles with equal spin, parity and baryon number (but different charge, isotopic spin and hypercharge), and is able to accommodate all established particles into the singlet(1), octet(8), decuplet(10) representations of the SU(3) symmetry group, while the Regge scheme puts together particles with equal parity and other internal quantum numbers into some angular momentum rotational bands (Regge trajectories). However more simple and universal empirical formulas for mass spectra are desirable for shaping up a dynamical model of strong interaction.

In searching for new mass formula, Shirkov[1] has analized the overwhelming majority of meson-meson and meson-baryon two-particle resonance and bound states in terms of the special energy variable

$$x=(M_{ik}^{res})^2-m_i^2-m_k^2.$$

Shirkov obtained one universal mass spectrum for meson-meson resonances, and two spectra for meson-baryon resonances. The results show that inside each J^P-multiplet x varies within 10%, and the mean values of $x(J^P)$ form a simple spectrum of a quasi-oscillator type. This is consistent with the harmonic oscillator model[2] of strong interaction in nuclear physics.

Here we introduce a new approach which leads to a more simple and universal mass formula for all baryon and resonance states. The essential feature of this method is the combination of SU(3) symmetry and Regge trajectories. Starting from Regge trajectories, the mass formula is obtained by investigating the dependence of the trajectory constants on other quantum numbers.

Phenomenologically almost all established particles and resonances can be fitted into certain trajectories which are approximately linear in the energy square variables with the following expression,

$$\alpha(s)=a+bs \qquad (1)$$

with a, and b constant for each trajectory, and varying with the trajectories. In general, they are functions of the quantum numbers of the corresponding trajectory, i. e. signature(τ), parity(P), and other internal quantum numbers. As is well known, b is experimentally found to be approximately a universal constant$\simeq 1$ (Bev)$^{-2}$ for all established meson and baryon trajectories. The main task is then reduced to a systematical analysis of the variation of a. This is done with the heip of SU(3) symmetry grouping and symmetry breaking effects in masses of hadrons.

For baryon spectra, we take Regge trajectories collected by Barger and Cline[3]. The values of a for those are found and plotted against hypercharge(Y) as shown in Fig. 1. A few remarks about the values taken for a should be made clear here. The value of

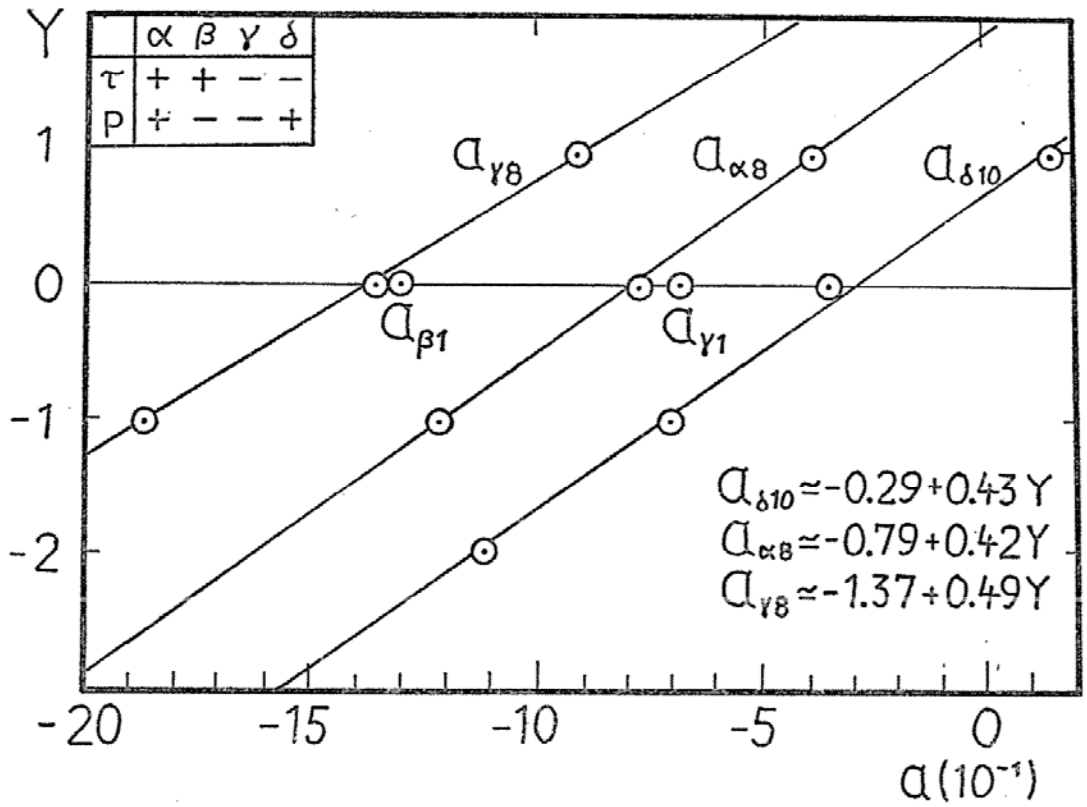

Fig. 1. The Mass Spectrum of Baryons

$a_{\alpha 8}$ at Y=0 takes the average of Λ_α and Σ_α trajectories. Values of $a_{\gamma 8}$ are determined by the extrapolations of the upper parts of γ-octet trajectories. If the extrapolations of the lower parts are taken instead, the curve of $a_{\gamma 8}$ will nearly coincide with $a_{\alpha 8}$, that is equally probable since only the N_γ-trajectory is better established. The figure shows three nearly parallel and almost equally spaced straight lines which can be approximately expressed as

$$a\simeq a_0+a_1 Y \qquad (2)$$

with constants $a_0\simeq 0.45$ and a_1 varying with different multiplets. The linear dependence on Y is the familiar SU(3) symmetry-breaking effect in masses of baryons. But the near parallelism and equal spacing is striking. In general, a_0 may be a function of τ, P, and the SU(3) indices of each multiplet. If two singlet trajectories Λ_β and Λ_γ are ignored, a universal function a_0 for octets and decuplets can be chosen to be

$$a_0\simeq -0.83+0.27\,(1-\tau)P \qquad (3)$$

From Eqs. (1)-(3), we get a universal mass spectrum for almost all baryons,

$$\alpha^{B}(s)\simeq -0.83+0.27(1-\tau)\mathrm{P}+0.45Y+s \qquad (4)$$

On the other hand, if $a_{\tau s}$ is taken to be coincident with a_{as} and the not so well defined $\alpha_{\beta 1}$-trajectory is ignored, then we can approximate a_0 as

$$a_0\simeq -0.55-0.25(\tau\mathrm{P}) \qquad (5)$$

Finally we would like to add some comments:

1. In above analysis, we have chosen a_0 to be independent of the SU(3) indices of each multiplet. The parity dependence of mass formula thus obtained contradicts the Gribov-MacDowell symmetry which predicts parity doublets. Should fermion conspiracy trajectories be confirmed by future experiments, a_0 has to be a function involving these indices in order to include these trajectories.

2. The same method can be applied to study the meson mass spectrum. Unfortunately present experimental data are insufficient for similar investigation; besides, the SU(3) symmetry breaking effects in mesons are rather complicated.

REFERENCES

(1) D. V. Shirkov, *USSR Academy of Sciences Preprint* Tφ-48 (1968).
(2) S. P. Elliot, *Rochester Preprint* NYO-2271 (1964)
(3) V. Barger and D. Cline, *Phys. Rev. 155*, 1792 (1967).

VISCOSITY OF ALKALI SILICATE GLASSES

URBAN E. SCHNAUS

I. INTRODUCTION

This is a report of viscosity measurements of twenty-two compositions of research glass, mainly binary alkali-silica materials: K_2O-, Na_2O-, and Li_2O with SiO_2. Samples of these materials, in the form of small beams about 7.0 cm long and having rectangular cross-section, 3.0 to 7.0 mm on a side, and of pellets in the form of right cylinders, 6.5 mm high and 6.5 mm in diameter, were sent to Taiwan from the Vitreous State Laboratory at the Catholic University of America in Washington, D. C., U. S. A., where these materials were being studied in research projects involving transmission of ultrasonic waves[1] and absorption and scattering of visible light[2] from a He-Ne laser.

In these researches the "fictive temperature" of the glass materials figures prominently. In the two publications just cited, the values used were taken from a publication by Poole in 1949[3]. While the data given in Poole's paper is remarkably good, it seemed advisable to make some further measurements on actual melt materials being used in the current research. That the situation is somewhat uncertain can be seen in the first two publications just cited, where Ref. 1 gives T_f=480°C for a 25% (mole per cent) K_2O binary silica glass, and 460°C for a 33% K_2O glass, while in Ref. 2 Table III gives values 732°K and 600°K (459°C and 427°C respectively) for the same points.

Ref. 3 has been used in other published work. In an article discussing the inadequacies of proposed viscosity theories, the authors[4], in referring to a theoretical model based on Poole's data for alkali silicate glasses, state: "the data they used was scanty and it was questionable whether the viscosity values represented measurements on glass in an equilibrium state."

In a private communication Schroeder states that "the proper value for the fictive temperature T_f for density fluctuations is $T(\eta=10^{13}$ poise$)-20$ C°". The major purpose of this research has

been to make measurements of this 10^{13} poise temperature for the 22 compositions that have been furnished. These data, with appropriate discussion, are in Section III of this report. An approach has been made to ascertain a corresponding temperature, $T_{f'}$, the fictive temperature for concentration fluctuations, which, according to this same communication from Schroder, is $T(\eta=10^6$ poise). Here the investigation has been somewhat less satisfactory, as the results in Section III show.

II. EXPERIMENTAL PROCEDURE

A. Preparation of Sample Materials

Each of the three references first cited above describe the method of preparation of samples. Possibly a source of difference in the findings of this paper from those of Ref. 3 can be found here. Poole's measurements were made by the fiber elongation method; the samples used were thin fibers, drawn by hand from the melt in a small platinum crucible in a Globar furnace. The fibers used in this method are between 0.5 and 1.0 mm in diameter and about 25 cm long. They had to be stored immediately after drawing in dessicators to avoid surface attack from atmospheric moisture.

The samples used in Ref. 2 were somewhat rough parallelepipeds 1 by 1 by 2 cm made by pouring the glass melt into molds. In making these melts reagent-grade components were thoroughly mixed and then melted in a Pt crucible in a Globar furnace also. "Each melt was stirred for at least five hours to achieve homogeneity and was then allowed to stand for an additional five hours at high temperature" for further fining (removal of small bubbles). From the high temperature (about 1400°C) melt the samples for the scattering experiments were poured. They were then annealed for six hours at a temperature close to T_f for the given composition, before being cooled to room temperature at initial rate about 2 C°/min, after which they were placed in dessicators.

Most of these materials are very hygroscopic. In the light-scattering experiments measurements were made with the samples kept in oil. The beams and pellets used for the viscosity measure-

ments here reported were made by cutting with diamond saws and grinders using a suitable coolant oil, after which they were coated with a heavier oil or with paraffin and packed with dessicant materials for mailing. Before use in the beam-bending or parallel-plate apparatus, this protective coating was removed with benzine and the necessary initial dimension measurements were made as quickly as possible while a slight amount of the coating still adhered to the surface.

B. Method of Measurement

1. Beam-bending

This method has come into significant use since 1962[5] and is now one of the methods approved by the ASTM for determining the annealing point (10^{13} poise) for vitreous materials. The version of this apparatus used in this investigation has been described in a previous issue of this journal[6]. Using samples of standard NBS glasses we are able to come regularly within 2 or 3 C° of the certificate values for the annealing point. This value, as now determined by the approved ASTM procedure, depends on measurements made at decreasing temperatures and is not a true equilibrium value. The Fulcher equations given on the NBS certificate sheets are derived from empirical data for viscosity 10^2 to 10^{12} poise and do not apply to this annealing point determination. As can be seen on the plots of log η vs 1000/T°K in Section III of this report, where direct viscosity measurements for binary and ternary glasses between 10^{10} and 10^{13} poise are shown, the circled 10^{13} poise points from measurements at decreasing temperature lie close to the line drawn from average values of viscosity measured at constant temperature.

To get sufficiently large clear samples for good beam-bending determinations is difficult with many of these compositions. No beams of K_2O-SiO_2 binary glass with less than 25 mole % alkali were available. A considerable number of beams of binary 15% Na_2O glass were bent, with rather strange results, as reported below.

2. The Parallel-plate Viscometer

This is a more recent development[7]. A simplified drawing of the apparatus is shown in Fig. 1. As in the beam-bending apparatus,

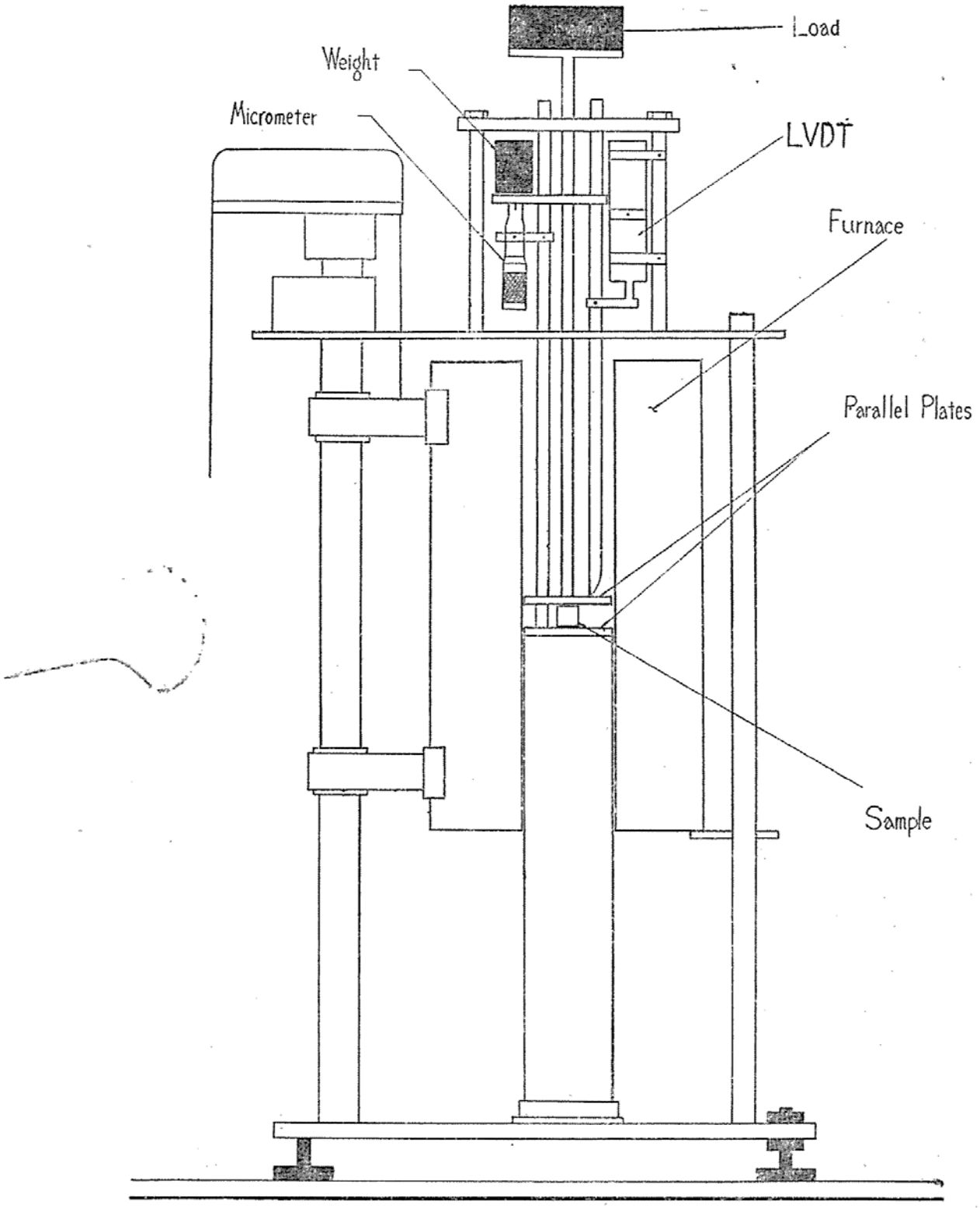

Fig. 1. Parallel plate viscometer

the central measuring element is a linear variable differential transformer (LVDT) which furnishes an electrical signal proportional to

a change in linear dimension. Here we measure the change in height of a solid right cylinder of glass as it is viscously deformed between two parallel plates in uniaxial vertical compression. The plates are of Inconel, a special alloy of high thermal conductivity suitable for such high-temperature work, used here in the form of discs about 4.0 cm in diameter and 3/4 cm thick. The lower disc is supported on a ceramic tube, and has drilled into one side a well for the measuring thermocouple. The upper plate has attached to its top a vertical metal push rod about 34 cm long, surmounted by a load pan. Two heavy vertical metal tubes, attached to the base and held parallel to the central ceramic tube, have at the top a guide frame for the loading rod and for fused silica transfer rods and the device for supporting and positioning the LVDT. A cylindrical electric furnace can be raised to surround the sample after it has been placed in position between the parallel plates.

In making a measurement the current in the heater windings is adjusted to raise the temperature of the sample at a rate about 5 C°/ minute. In some glasses ("long" glasses) the softening process can be followed through a temperature change of more than 300 C°. A "run" requires about two hours and ends with the cylindrical pellet changed to a disc often less than 0.5 mm thick and over 2.5 cm in diameter.

The equation for determining viscosity by this method (Ref. 7, pg. 595) is

$$\eta = \frac{2\pi \mathrm{M}gh^5}{3\mathrm{V}(2\pi h^3 + \mathrm{V})dh/dt}$$

where Mg is the applied vertical load force in dynes, h the height of the cylinder at the given temperature and load point, in cm; V is the volume of the sample cm^3, and dh/dt the time rate of change of h at the given point. Minimum M for the apparatus used is 209 gm, which is the mass of the top plate, load rod, and associated elements of the LVDT system. Suitable change in load must be made in the course of a run to maintain reasonably constant applied stress; loads between 400 and 1800 gm have been found satisfactory with the

apparatus here used. V is considered to be constant, determined from the original dimensions of the sample. The values of h and of dh/dt are obtained from the recorder chart and from the calibration constant of the LVDT. Thickness h for a given point is determined from the final specimen thickness and the change in height shown by the total deflection of the recorder pen. The slope of the tangent line to the curve on the recorder chart at a given temperature point is used to find dh/dt.

An x-y recorder with a variable-speed paper take-up has been found useful in these investigations. A programmable electronic calculator is also very convenient for this work, since temperature points are indicated at intervals 5 to 10 C° apart. Over 100 sample pellets were used in this work; each recorder chart has between 20 and 40 points.*

An inherent difficulty in the use of the parallel-plate viscometer must be mentioned. As is evident from the equation used for calculating the viscosity, accuracy of measurement of h is very critical, since h^5 and h^3 appear in the basic equation. Check runs using NBS No. 710 standard glass were made throughout the course of the work, and agreement with standard values was good between 10^7 and 10^{10} poise (760°C to 630°C), with best agreement around the 10^7 poise point, our η vs T plots tending to give lower temperature values at higher viscosities and higher temperature values at lower viscosities. For the determinations here treated the average accuracy and precision of ±4 C° given in Ref. 7 (pg. 596) for this method would not apply: ±10 C° would be a more realistic figure for these investigations. However, in all cases the η vs T plots are sufficiently smooth and regular, and the usefulness of this method of determining

* Grateful recognition is due here to several FuJen University Senior Physics Major students who helped with this work. Calvin Lin measured most of the recorder charts from the parallel-plate apparatus to determine Δh and $\Delta h/\Delta t$; he also made sample beams and pellets of NBS standard glass with a small diamond saw. Measurements and calculations for most of the beam-bending work was done by Frank Chen. The black-and-white drawings for figures in this report were made by Jeff Lin and Halliday Ong; Ong also made water-color diagrams of the beam-bending and the parallel-plate apparatus for wall posters in our laboratory.

the viscosity of relatively large samples of vitreous materials has been amply demonstrated.

III. RESULTS

A. General Considerations

A widely-used method of treating viscosity data such as these reported here is to fit the data to an equation of the form: $\log_{10} \eta = a + \frac{b(10)^3}{T°K}$ and to display the points on a plot with η as ordinate and $10^3/T°K$ as abscissa (Arrenhius plot). This linear form has been found suitable for many kinds of glass and is used in Ref. 3 for the 11 binary and 10 ternary alkali silica compositions there studied. Values of a and b were determined for each of these compositions by the method of least squares. This method is satisfactory for most of the compositions in the range of viscosity 10^9 to 10^{12} poise. It is also fairly suitable for the range over which beam-bending measurements can be most conveniently made, from 10^{10} to 10^{13} poise.

At higher temperatures and over wider ranges of viscosity such linear equations are not suitable; in the widely-used Fulcher equation discussed in Ref. 4 (pg. 1889), a third arbitrary constant is added to make the denominator of the fraction $=(T-T_0)$. For the NBS standard glasses No. 710 and No. 711 this gives a good fit for viscosities between 10^2 and 10^{12} poise. In the work here reported no least-square analysis has been carried out to determine any of these arbitrary constants. But the Arrenhius form has been retained in Figs. 2 through 6 for convenient comparision.

In most cases it was not possible to carry the parallel-plate measurements down to 10^6 viscosity, and devitrification was observed throughout the samples when the runs were completed. On the plots the lines have not been extended below the approximate point where change in vertical dimension stopped with increasing temperature. Obviously the point at which devitrification begins is at some lower temperature which cannot be determined from our data. In all cases the slope becomes much smaller than that of the No. 710 standard reference glass.

The plots have been made using average values gotten from runs of 4 to 10 samples of each composition. For the parallel-plate runs the value of η for points 10 C° apart was read from the original η vs T plots for each sample of the 15 different compositions. The solid line of the plots was drawn from these average values, and linear extensions to the 10^{13} poise line have been made for most compositions for which no beam-bending data are available.

For the 8 different compositions of beams that were studied the scatter inherent in constant-temperature beam-bending measurements did not warrant a direct-averaging process. However, the η vs T data for ten or more determinations for each composition showed adequate uniformity to make a graphical linear evaluation. This has been translated to the Arrenhius plots to give the higher values of η shown in lines. The circled points on the 10^{13} poise line are mean values of 5 or more determinations of the annealing point by the ASTM decreasing-temperature procedure.

Agreement with previously reported work is best for the Na_2O binary compositions of higher soda concentration; log η isokoms that can be drawn from the data here reported are quite similar to those given in Ref. 3. With lower concentrations of soda or potash the differences from previously reported fiber-elongation data are considerably greater. And for binary potash-silica compositions having less than 20 mole % K_2O strange anomalies appear.

B. Discussion

1. K_2O Binary Compositions

a) Figure 2 represents the widest range of measurement for a single composition in this report. Both beams and pellets of 33% K_2O glass (glass b) were provided, and the measurements of 5 pellets and 5 beams of this composition show good internal agreement. But the lines for the two different methods of measurement do not join smoothly. It may be worth noting that Poole (Ref. 3, p. 231 footnote) found a comparable break in his line for a similar composition (his glass No. 11, 34% K_2O) and chooses to take the slope of his line from the two lower points. He says: "The dashed line represents the more probable slope. This is partially justified because of the ex-

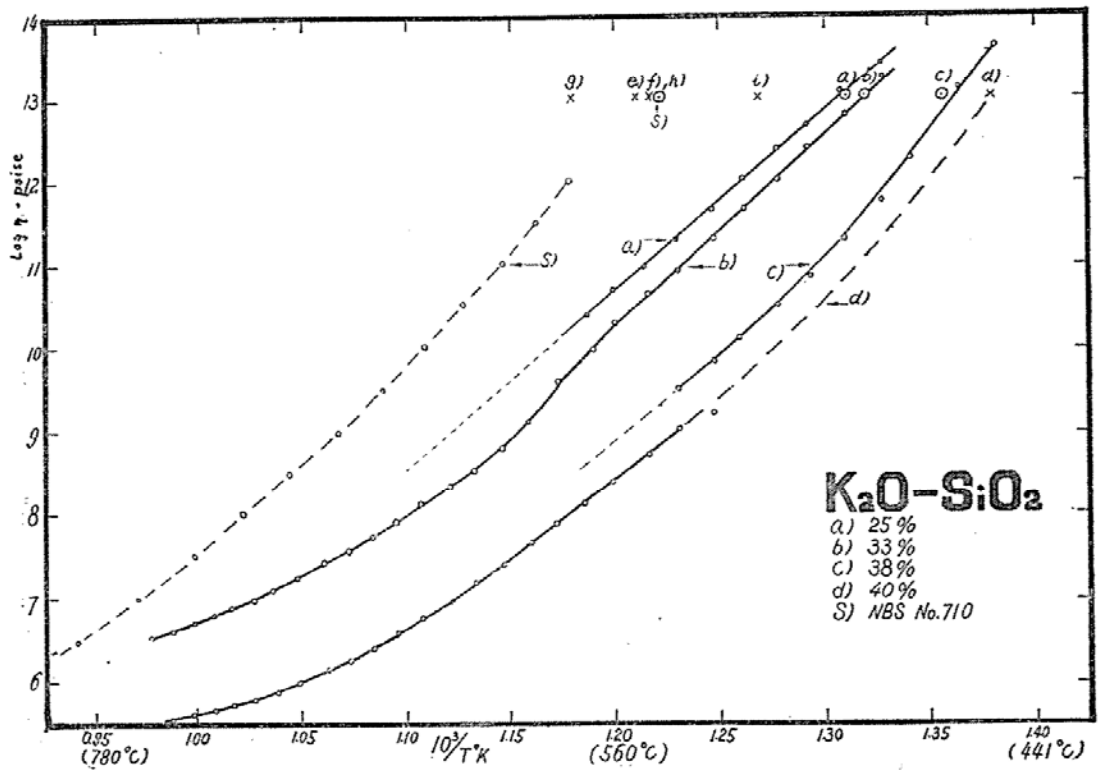

Fig. 2. Viscosity vs $10^3/T°K$ K_2O-SiO_2: 25-40 mol %

treme hygroscopicity of Glass No. 11 that makes such measurements difficult." Many others of these compositions show much the same hygroscopicity, and the lines for pellets of 38% potash and beams of 40% potash do not show this pronounced difference of slope around this point.

Indicated values of the 10^{13} poise point are as follows: a) 25%=491°C, b) 33%=484°C, c) 38%=463°C; the corresponding values from Ref. 3 are: 24.5%=474°, 30.2%=450°, and 24.0%=436°. The points g, e, f, h, and i are straight-line extrapolations from the parallel-plate data of Fig. 3, and point d has been determined by extrapolation from parallel-plate data on this figure. Broken line s represents certificate values for standard NBS No. 710 glass that was used for check runs throughout throught this work and so is indicated on this and succeeding figures.

b) Figure 3 summarizes parallel-plate data for compositions less than 25 mole % K_2O. Compositions less than 8% K_2O are not reported in Ref. 3. In a private communication Schroeder states that

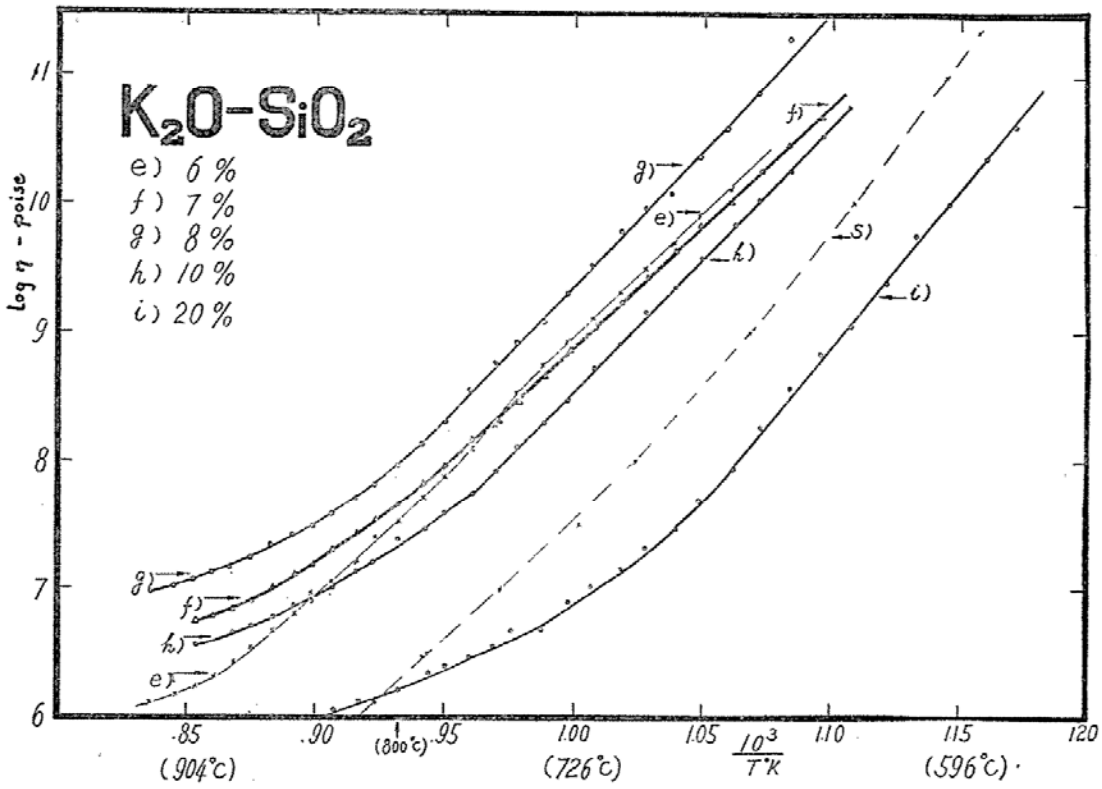

Fig. 3. Viscosity vs $10^3/T°K$ K_2O-SiO_2: 6–20 mol %

his work with these compositions contains the first published report on phase separation in K_2O-SiO_2 binary glasses. Certainly we see here distinctively non-additive behavior of K_2O as a softening component in silica melts: increasing concentration of potash from 6 to 8% results in decreasing softness of glass. A log η isokom ($\eta=10^{10}$ poise) is indicated in Figure 7; it is evident that in this small range of compositions the simple continuity of Poole's isokom (Ref. 3, Fig. 2 p. 231) is not followed.

The usefulness of viscosity measurements for determining the critical immiscibility temperature in phase-separated glasses has been demonstrated[8], and the unexpected differences in these plots seem to point to some phase-separation in this case. Considerable theoretical and experimental work along these lines is being done, especially at the National Bureau of Standards in Washington[9,10], working mainly with borosilicate glasses having four or more components. The results here reported indicate that similar studies with

these binary glasses may be of interest.*

In Ref. 2 (Fig. 4, p. 213) a smooth extrapolation is made on the plot of T_f (10^{13} poise) vs concentration from the 8% point to the 1600°K point for pure SiO_2. It is evident that locating values from our Fig. 2, points g, e, f, h and i will not give such a smooth line. That such extrapolations are not always justifiable is indicated in figure 4.

2. Na_2O Binary Compositions

The work done with all available samples of binary Na_2O silicate glass is summarized in Figure 4. Here the continuity between parallel-plate and beam bending data for the 33% (glass 1) composition is much better than for the corresponding binary potash glass. Also, the agreement of the 10^{13} poise points: k, l, and m, with the corresponding ones of Ref. 3 is quite good.

However, for compositions having 15% and 25% Na_2O (glass j and glass k) the results are much different. Phase separation for composition j has been reported[11]. The behavior of beam samples used gives evidence of change in macroscopic elastic properties. Efforts to get consistent measurements of normal viscous deflection under load at equilibrium temperatures between 500°C and 800°C proved futile. Several of the beams turned opalescent under this heat treatment; the appearance of opalescence is the oldest and most widely used method for detecting liquid-liquid immiscibility.† A pellet cut from one of the beams that had turned opalescent was used as a sample in the parallel-plate apparatus and showed much different viscosity from the original clear samples.

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* Conclusion (1) of Ref. 10 p. 1170 states: "that the viscosity of an oxide system that decomposes into two immiscible liquids at a critical temperature increases anomalously as the critical temperature is approached from above." Ref. 8 (footnote, p. 299) defines "critical immiscibility temperature" as the upper phase-separation temperature of a given glass composition. In the measurements with the parallel-plate apparatus the approach is from below while in the ASTM procedure for determining the annealing point the approach is from above in temperature. How close the "knee" (about 680°C) of the curve in the 15% Na_2O binary composition of Fig. 4 is to the critical immiscibility temperature for this composition remains to be determined by other investigations.

† Ref. 8, p. 299.

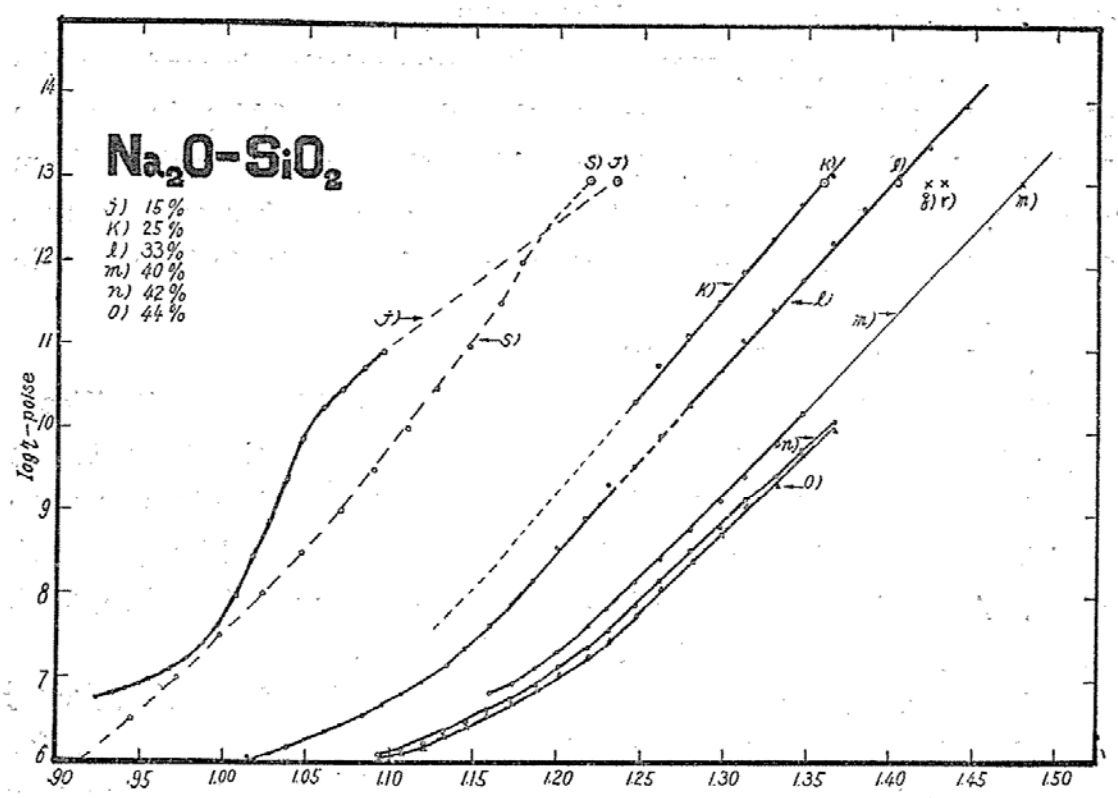

Fig. 4. Viscosity vs $10^3/T°K$ K_2O-SiO_2 glass

While the five sample pellets of glass j gave η vs T plots of satisfactory uniformity, the slope of the straight line portion of the curve is much different from that gotten by fiber elongation measurements reported in Ref. 3. And, while efforts to get equilibrium viscosity measurements by beam-bending proved futile, measurements using the decreasing-temperature method showed reasonable consistency; hence the j point on the 10^{13} poise line has been indicated for this composition, though the broken line extension of the parallel-plate data to this point is open to question. We seem to have here another example of the "conflicting results" mentioned in a recent publication[12] dealing with these matters.

The difference between the 42% and the 44% binary soda compositions is probably not significant; but there is a significant difference between the 40% and the 44% composition. Comparision of Fig. 4 with Fig. 2 shows soda to be a more effective softening additive than potash in binary silicate compositions: the annealing point values (10^{13} poise) are 463°C and 440°C respectively for the 25% and

the 33% binary soda compositions, while for the respective binary potash compositions the temperature values are 491°C and 484°C.

3. Li_2O Binary Compositions

A few sample pellets of binary lithium silicate glass were supplied; Figure 5 is a summary of parallel-plate data from ten runs of three different compositions. The results show satisfactory regularity, and extrapolation to the 10^{13} poise line gives the annealing point about 430°C for the 33 mole % Li_2O binary composition, with correspondingly higher and lower values for the 30% and the 36% compositions.

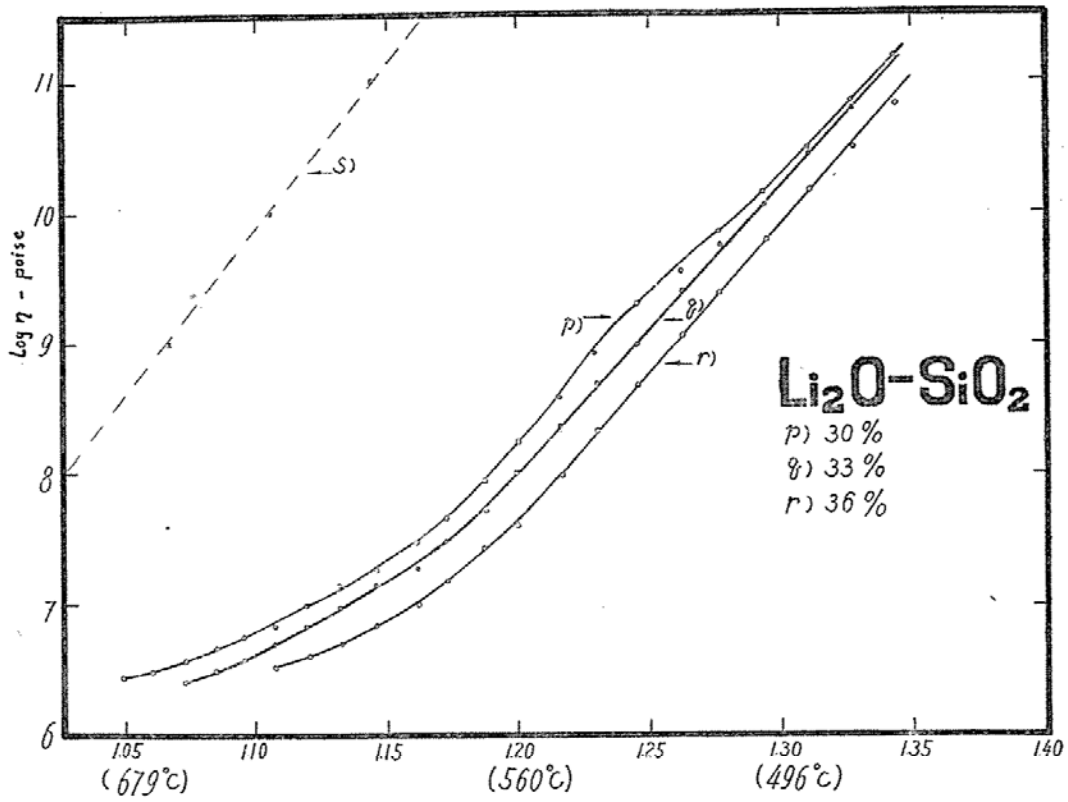

Fig. 5. Viscosity vs $10^3/T°K$ Li_2O-SiO_2 glass

Earlier work with a few 33% Li_2O-SiO_2 beams resulted in much the same kind of conflicting results as with the 15% Na_2O binary composition. The beams became cloudy, and in some cases devitrified; runs at equilibrium temperatures showed poor consistency, though runs using the ASTM decreasing-temperature procedure showed

considerably better regularity. A series of 14 determinations by this method shows a spread of 450°C to 487°C with mean value 474°C for the 10^{13} poise point. Extrapolation of the parallel-plate line for this 33% Li_2O composition, glass p, is shown on the preceding Fig. 4; the value so obtained for the 10^{13} poise point is 430°C. A plot of the badly-scattered equilibrium temperature values for the composition gives a line that puts the 10^{13} poise temperature about 450°C.

4. Ternary Compositions of 75% SiO_2 with Na_2O and K_2O

Beam-bending measurements of ternary compositions containing 75% SiO_2 are summarized in Figure 6. The cumulative, but non-critical, softening activity of Na_2O plus K_2O appears here, where the three compositions: u, v, and w show little significant difference above 10^{12} poise in determinations at equilibrium temperatures, and the 10^{13} poise point determined by the decreasing-temperature method is practically the same as that from equilibrium temperature measurements. But the 430°C temperature so obtained is over 30 C° less than for binary 25% Na_2O beams. The other one of the four ternary

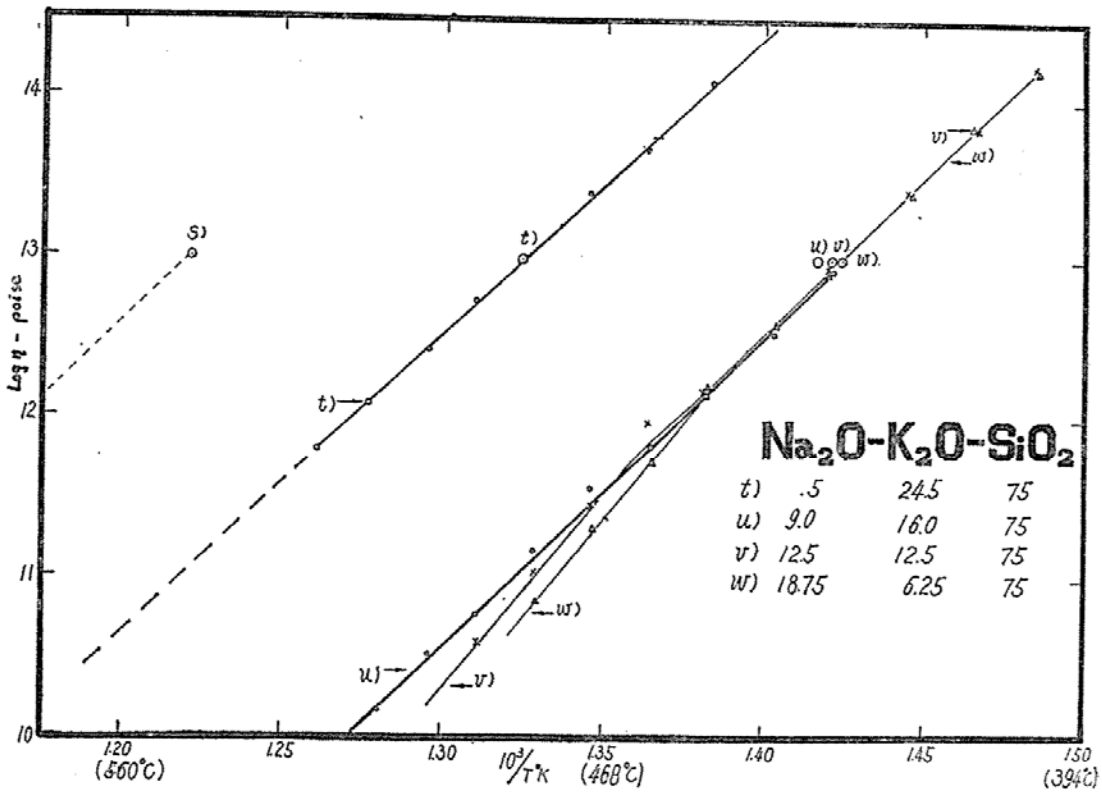

Fig. 6. Viscosity vs $10^3/T°K$ Na_2O-K_2O-SiO_2 glass

compositions, glass t on Fig. 6, has annealing point temperature about 7 C° lower (484° as compared to 491°) than for pure 25% K_2O binary glass: "doping" with ½% Na_2O produces a significantly lower annealing temperature.

While the overall agreement of these ternary 75% SiO_2 compositions with the data reported in Ref. 3 is fairly good, it may be worth noting that our variations with temperature for the different compositions are in the opposite direction from those reported by Poole. The data here reported show that differences in viscosity with change in concentration begin to appear only as temperature increases; the 1949 data show two of the compositions, approximately equivalent to glass t and glass u of Fig. 6, coinciding at temperatures above 10^{10} poise and diverging at lower temperatures. Assuming that the fiber elongation measurements, like our beam-bending ones, have been made at equilibrium temperatures, the difference in behavior appears to be connected with sample size and treatment. It is to be regretted that no pellets of these compositions were available to follow the lines down to lower viscosities.

IV. SUMMARY AND CONCLUSIONS

The results here reported are somewhat incomplete and of a preliminary nature. As is evident from the Introduction, most of the samples furnished came as a sort of by-product of other research work being done elsewhere. But even with the evident gaps and the broken-line extrapolations on Figures 2 through 6, where all the measurements are summarized, some useful data has been obtained for those who provided the samples, and some new information has been gotten that may be of interest to others working in this field.

This work has been done with rather simple equipment in a part of the world where facilities for making research glasses are severely limited. None the less, a few significant points that appear from this investigation may be mentioned:

1) Phase separation on a macroscopic scale in binary alkali silica glasses may become masked by the methods of sample preparation and measurement in viscosity determinations. The evident differences

between the data here given and those found in Ref. 3 for binary compounds less than 20% Na_2O and K_2O give instances of this.

2) Uniform increase of K_2O concentration between 5% and 9% in binary potash-silica glasses does not result in a corresponding monotonic increase in "softness" of the glass. Figure 7, a log η isokom ($\eta=10^{10}$ poise) for compositions of Fig. 3 over a limited range (615°C

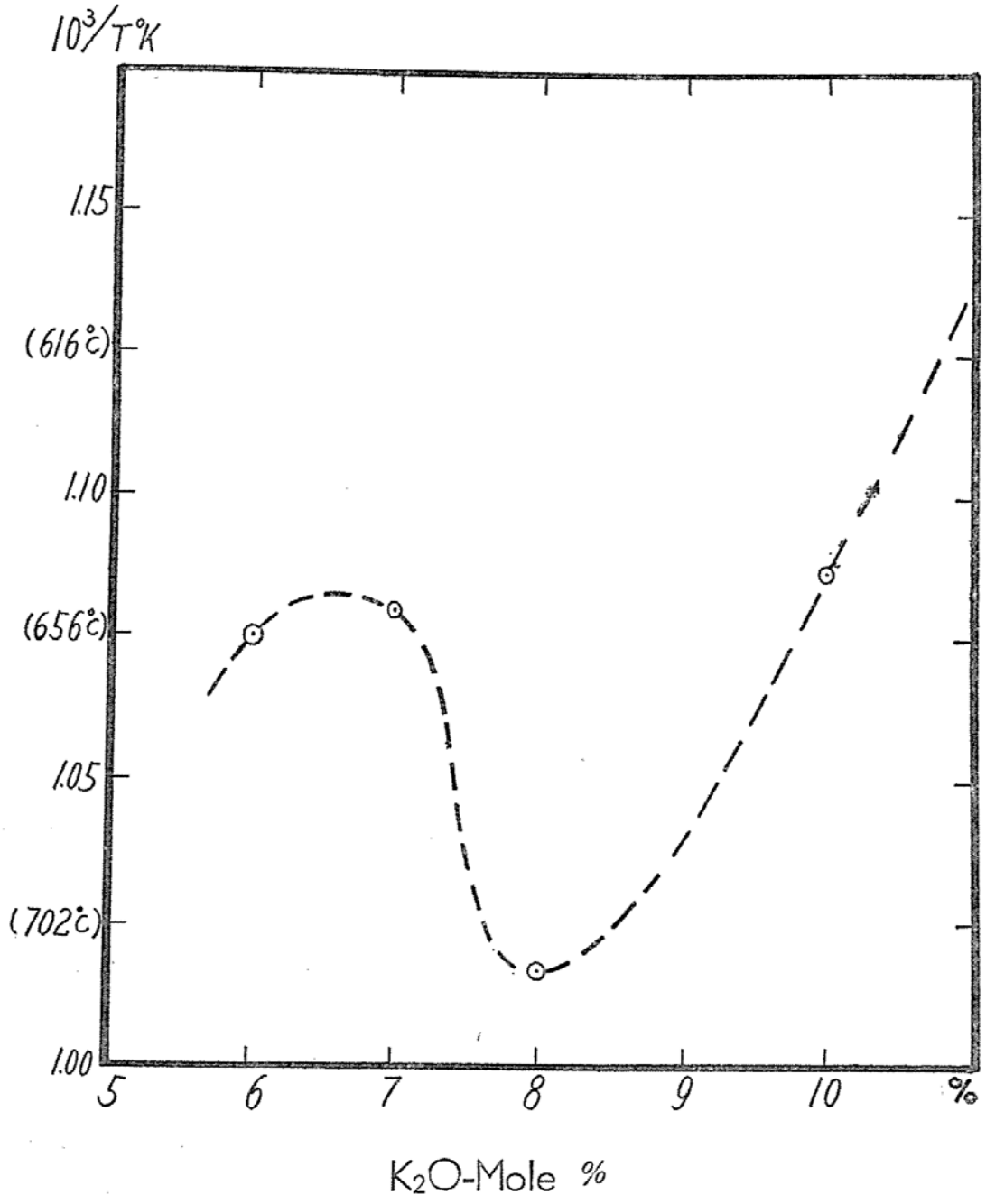

Fig. 7. $10^3/T°K$ vs mol % K_2O: K_2O-SiO_2 glass

to 725°C and 6% to 10% K_2O concentration), shows a distinct "dip" around the 8% K_2O point rather than the uniform increase that might be expected. The doctoral dissertation of Schroeder recently published has some significant discussion of this point.[13]

3) In ternary compounds: (A) Na_2O+(B) K_2O+75 mole % SiO_2 viscosity measurements by beam-bending show little difference for A between 9 and 18 and B between 6 and 16 at viscosities greater than $10^{12.5}$ poise. Differences between these ternary compounds do show up for η less than $10^{11.5}$ poise.

If sample materials can be obtained, work along these lines will continue and will be reported in the next issue of FU JEN STUDIES.

ACKNOWLEDGMENTS

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2. A continuing grant (62-03-0029) of the National Science Council, Republic of China, has assisted in providing supplies and auxiliary apparatus.

3. Mr. Albert Napolitano of the Inorganic Glass Section at the National Bureau of Standards in Washington has furnished generous amounts of Standard Research Glasses and other materials for this work. The author acknowledges with pleasure the courtesy shown him by Mr. Napolitano, Drs. Simmons and Haller, Mrs. Mills, and others of the Section when he visited them at the Bureau in January 1974.

REFERENCES

(1) Laberge, N. L., Vasilescu, V. V., Montrose, C. J., and Macedo, P. B., "Equilibrium Compressibilities and Density Fluctuations in K_2O-SiO_2 Glasses" *J. Amer. Ceram. Soc.*, **56**[10] 506-509 (1973).

(2) Schroeder, J., Mohr, R., Macedo, P. B., and Montrose, C. J., "Rayleigh and Brillouin Scattering in K_2O-SiO_2 Glasses"; *J. Amer. Ceram. Soc.*, **56**[10] 510-514 (1973).

(3) Poole, J. P., "Low-Temperature Viscosity of Alkali Silicate Glasses"; *J. Amer. Ceram. Soc.*, **32**[7] 230-233 (1949).

(4) Macedo, P. B., and Napolitano, A., "Inadequacies of Viscosity Theories for B_2O_3" *J. Chem. Phys.*, **49**[4] 1887-1895 (1968).

(5) Hagy, H. E., "Experimental Evaluation of Beam-bending Method of Determining Glass Viscosities in the Range 10^6 to 10^{15} Poises"; *J. Amer. Ceram. Soc.*, **46**[2] 93-97 (1963).

(6) Schnaus, U. E., "Thermal Analysis Studies: Measurement of Viscosity of Glass"; *Fu Jen Studies*, **6** 75-94 (1973).

(7) Fontana, E. H., "A Versatile Parallel-plate Viscometer for Glass Viscosity Measurements to 1000°C"; *Amer. Ceram. Soc. Bul.*, **49**[6] 594-597 (1970).

(8) Haller, W. K., Simmons, J. H., and Napolitano, A., "Viscosity-drift technique for Determination of Critical Liquid-Liquid Immiscibility Temperature of Glasses", *J. Amer. Ceram. Soc.* **54**[6] 299-302 (1971).

(9) Simmons, J. H., Macedo, P. B., Napolitano, A., and Haller, W. K., "Investigations of Liquid Phase Transitions in Oxide Melts by Viscosity Measurements", *Disc. Faraday Soc.* No. 50 (1970).

(10) Simmons, J. H., Napolitano, A., and Macedo, P. B., "Supercritical Viscosity Anomaly in Oxide Mixtures", *J. Chem Phys.* **53**[3] 1165-1170 (1970).

(11) Porai-Koshits, E. A., and Averjanov, V. I., "Primary and Secondary Phase Separation of Sodium Silicate Glasses", *J. Non-Cryst. Solids*, **1**[1] 29-38 (1968).

(12) Simmons, J. H., Mills, S. A., Napolitano, A., Blackburn, D. H., and Haller, W. K., "Investigation of Viscous Flow in Glass During Phase Separation", *Nat. Bur. of Standards Tech. Note 792* (Sept. 1973).

(13) Schroeder, J., "*Rayleigh and Brillouin Scattering in Amorphous Solids: Silicate Glasses*", The Catholic University of America, 1974. In Ch. VI, Sect. C, pages 279-304 are of particular interest. Data here reported for binary K_2O compositions are included in Table VI. 1, p. 292.

PHOTOLYSIS OF PENTAFLUOROBENZENE

SO-LAN LEM TSENG

I. INTRODUCTION

In the past two decades, the photophysical and the photochemical behaviors of benzene and its fluorine derivatives have been interesting to chemists. The fluorescence yields and the triplet yields of all fluorinated benzenes, except 1, 2, 3–trifluorobenzene, were reported.[1]

The spectrofluorometric study on vapor phase pentafluorobenzene (PFB) at a variety of exciting wavelengths starting at 254 nm and extending to 278 nm showed the fluorescence of PFB to be very weak. The fluorescent yield was wavelength dependent. At 266 nm the quantum yield was 0.020 and independent of the pressure of PFB. Addition of inert vapor, e. g., cyclohexane, carbon dioxide or cis-butene–2, caused an increase in the fluorescent yield. Among all of the foreign gases used, cis-butene–2 was the most effective one to enhance the fluorescence of PFB.[2] A prediction of 1800 torr of cis-butene–2 will save the excited PFB molecules for fluorescence. At that point, the quantum yield will be tripled.

Like the fluorescent yield, the triplet yield of PFB was low. By use of sensitized emission of biacetyl (BiA), when the pressure of both PFB and BiA were 20 torr, at 266 nm, the triplet yield was 0.05. The triplet yield increased with increasing pressure of BiA.[2] This may be explained by the short triplet lifetime of PFB, which caused difficulties in the determination of the triplet yield by using the indirect methods of measurement used. By comparing the triplet lifetime of PFB with that of benzene, the pressure of BiA required to pick up all triplet energy from PFB will be 110 torr, but only 0.15 torr will be enough for benzene.

The large magnitude of the quantum deficit may be due to one or both of the following causes:

(a) Radiationless transitions to the ground state including internal conversion from first excited singlet to ground, and intersystem crossing from first excited triplet to ground. Although the evidence

is not conclusive, it is believed that radiationless transitions may occur via a thermodynamically unstable isomeric intermediate.[3,4]

(b) Photoproduct formation.

Benzvalene and fulvene have been found in photolysed benzene.[5-8] Both physical and chemical methods indicated the formation of isomers when benzene was excited in the region of 240–270 nm. Photoisomerization of trifluorobenzenes[9] and hexafluorobenzene[10] have also been reported.

The photoisomerization of PFB were reported by Camaggi and Gozzo[11] and Ratajczak.[12,13] Two products were found to be 2, 3, 4, 5, 6–pentafluorobicyclo [2, 2, 0] hexa–2, 5 diene and 1, 3, 4, 5, 6,-pentafluorobicyclo [2, 2, 0] hexa–2, 5 diene. They were characterized by infrared, n. m. r., and mass spectroscopy. The quantum yields of the photoisomers would provide the information on the fate of the excited molecules of PFB.

II. EXPERIMENT

The PFB used in this study was obtained from Peninsular Chemresearch Incorporated. Its purity was determined by use of gas chromatography with a 10–foot, 1/4″ diameter, 10% Tricresylphosphate on Chromosorb W (80–100 mesh) column. The percentage of contamination was found to be no more than ~0.1%. It was therefore, used without further purification.

The ACS reagent grade glacial acetic acid used had the minimum concentration of 99.8%. High purity grade nitrogen with ~3 ppm of oxygen content were used without further purification.

The experiment was carried out in a conventional grease- and mercury-free high vacuum line. Pressure measurements were performed by means of a spiral gauge. Mixing was effected by circulating the vapors through a flow-through cell with an all-glass, electromagnetically driven, piston circulating pump. The flow-through cell consisted of an inner cylinder of quartz (diameter 3.5 cm) concentric with the outer jacket of pyrex (diameter 10 cm). The inner cylinder was tappered to a spiral of 8 mm tubing near the bottom of the cell. The internal volume of the cell was found to be 2.54 liter.

The light source used was a coiled low pressure mercury lamp. During a run, the lamp was suspended inside the cylinder. To remove the 184.9 nm mercury emission, 52% aqueous solution of acetic acid was passed continuously through the cylinder. This solution cut off totally the light below 245 nm. At 253.7 nm, the transmittance was approximately 53%. The potassium ferrioxalate actinometer[14] was used to measure the lamp intensity. With 52% acetic acid solution in the cylinder, the reaction vessel received 2.1×10^{21} quanta per hour. A continuous flow of 15 ml/min of the filter solution was maintained during the photolysed period.

For the purpose of preventing decomposition of excited PFB molecules at high vibrational levels, nitrogen was employed. After photolysis, the products were frozen and nitrogen was pumped away before separation and further analysis.

The photolysed material was separated by chromatography. The sample was introduced in the Hewlett-Packard Model 7260A GC by a homemade heated ampoule crusher. The separation was effected by use of a 10-foot, 3/8″ diameter column, packed with 20% Diisodecyl Phthalate on Chromosorb A (60-80 mesh).

III. RESULT

It was found that irradiation of PFB with light of wavelength 253.7 nm produced small amounts of two major products. The yields of the two products were proportional to the length of irradiation, i. e., proportional to the intensity absorbed, up to 12 hours. As shown in Fig. 1, the ratio of the amounts of the two photo products was 1:5. The result of quantum yield versus irradiation time was plotted and is shown graphically in Fig. 2. The quantum yield increased with increasing time of irradiation until approximately five hours. With longer irradiation time, the quantum yields of both products decreased slightly. This could be due to the difference of the extinction coefficient between the products and the parent molecule. The quantum yields of photo products varied with the pressure of nitrogen used (from 100 torr to 1 atm) during the photolysis.

The yields of the photo products were extremely low. The

structures of the products were assigned by comparison of the observed chromatographic retention times and by analogy with the

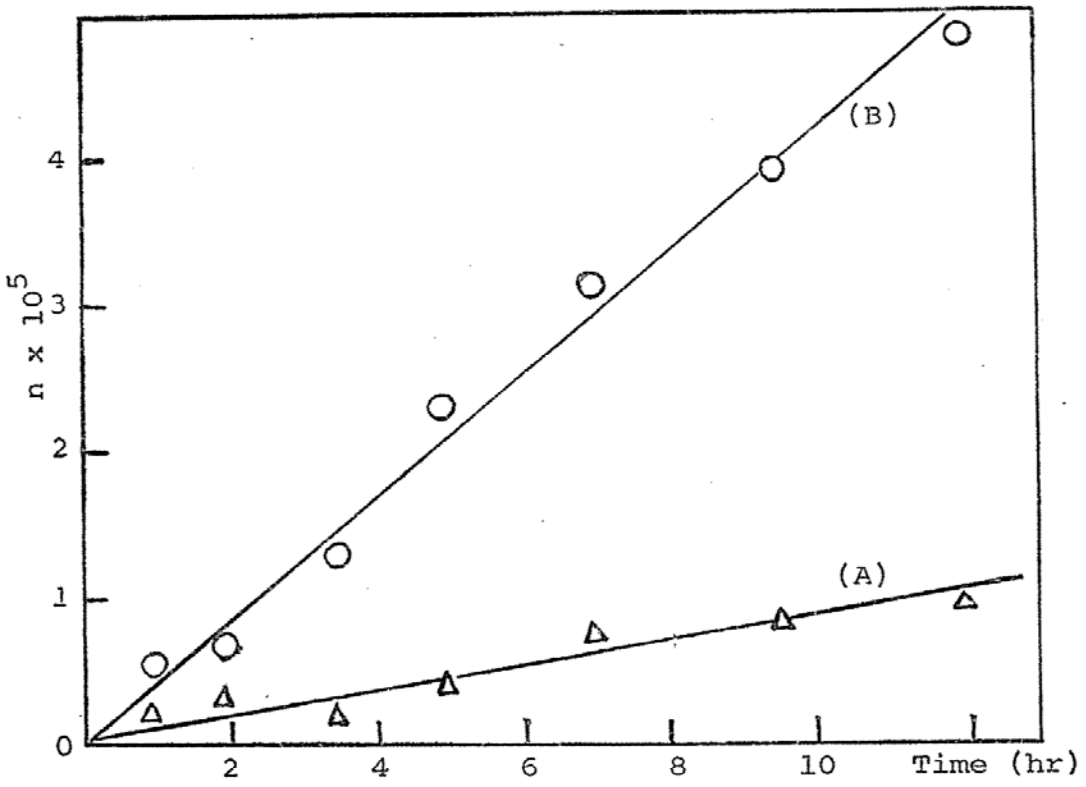


Fig. 1. Number of Moles of Photo-product versus Irradiation Time

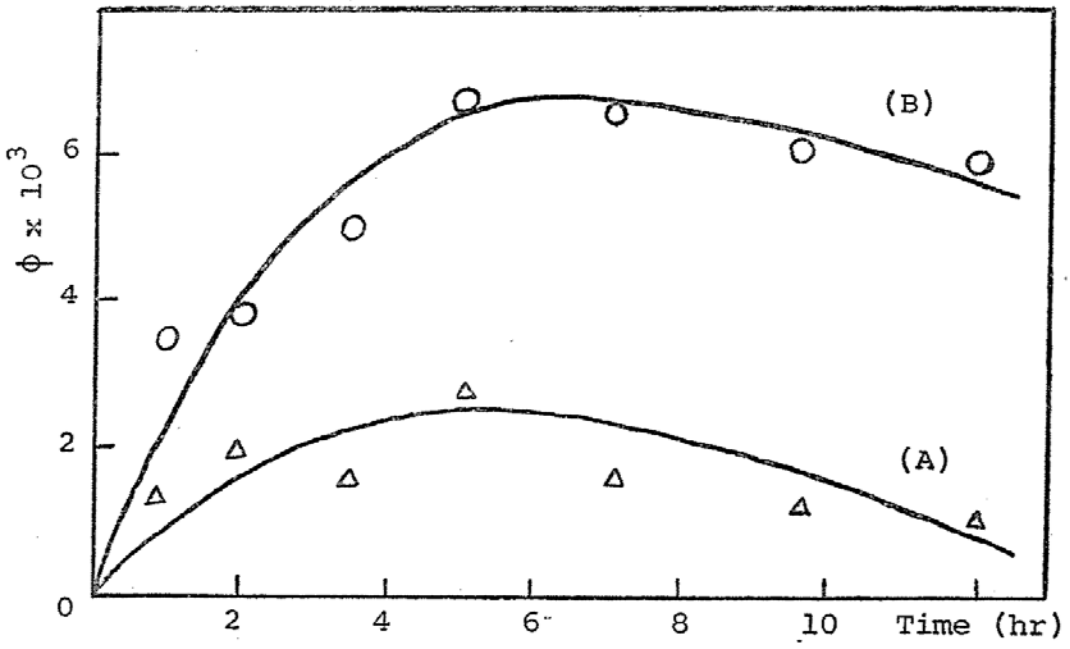
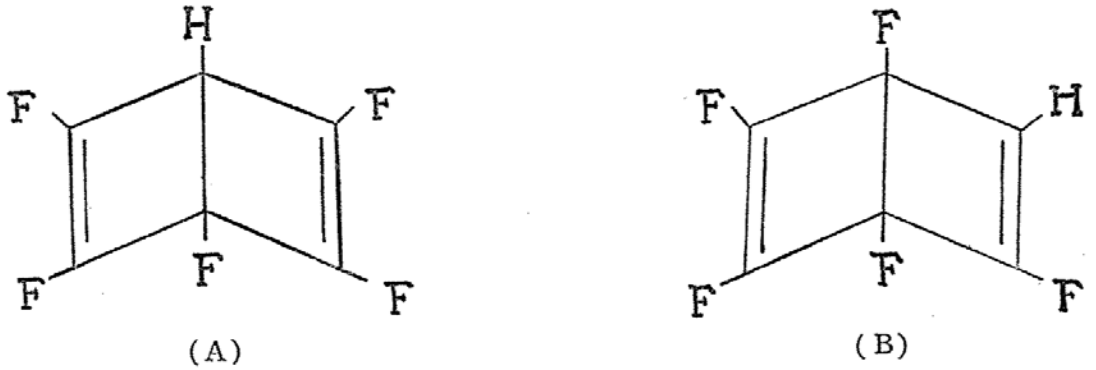


Fig. 2. Quantum Yield versus Irradiation Time

compounds studied by Ratajczak *et al.*[12,13] The one that had shorter retention could probably correspond to structure (A) of Fig. 3, i.e., 2, 3, 4, 5, 6–pentafluorobicyclo [2, 2, 0] hexa–2, 5 diene, while the one with longer retention could probably correspond to structure (B), i.e., 1, 3, 4, 5, 6-pentafluorobicyclo [2, 2, 0] hexa–2, 5 diene.



(A) (B)

Fig. 3. Structures of Photo-products

IV. DISCUSSION

A kinetic scheme may be proposed for the photo product formation:

$$PFB + h\nu \rightarrow {}^1PFB_v^I \quad (1)$$

$${}^1PFB_v^I + M \rightarrow {}^1PFB_a^I + M \quad (2)$$

$${}^1PFB_v^I \rightarrow A_v \quad (3)$$

$${}^1PFB_a^I (+PFB) \rightarrow PFB (+PFB) \quad (4)$$

$$A_v \rightarrow PFB \quad (5)$$

$$A_v + M \rightarrow A + M \quad (6)$$

where A is a product. The Arabic numeral denotes spin multiplicity while the Roman numeral denotes first excited state. The subscript "*v*" indicates high vibrationally excited species, and "*a*" indicates lower vibrational species including the ground vibrational level. M represents nitrogen.

The scheme proposed was based on the following assumptions:

(a) Absorption of irradiation led to a vibrationally excited singlet electronic state. Since the exciting light had a fairly wide band pass and there was a natural broadening of vibrational states,

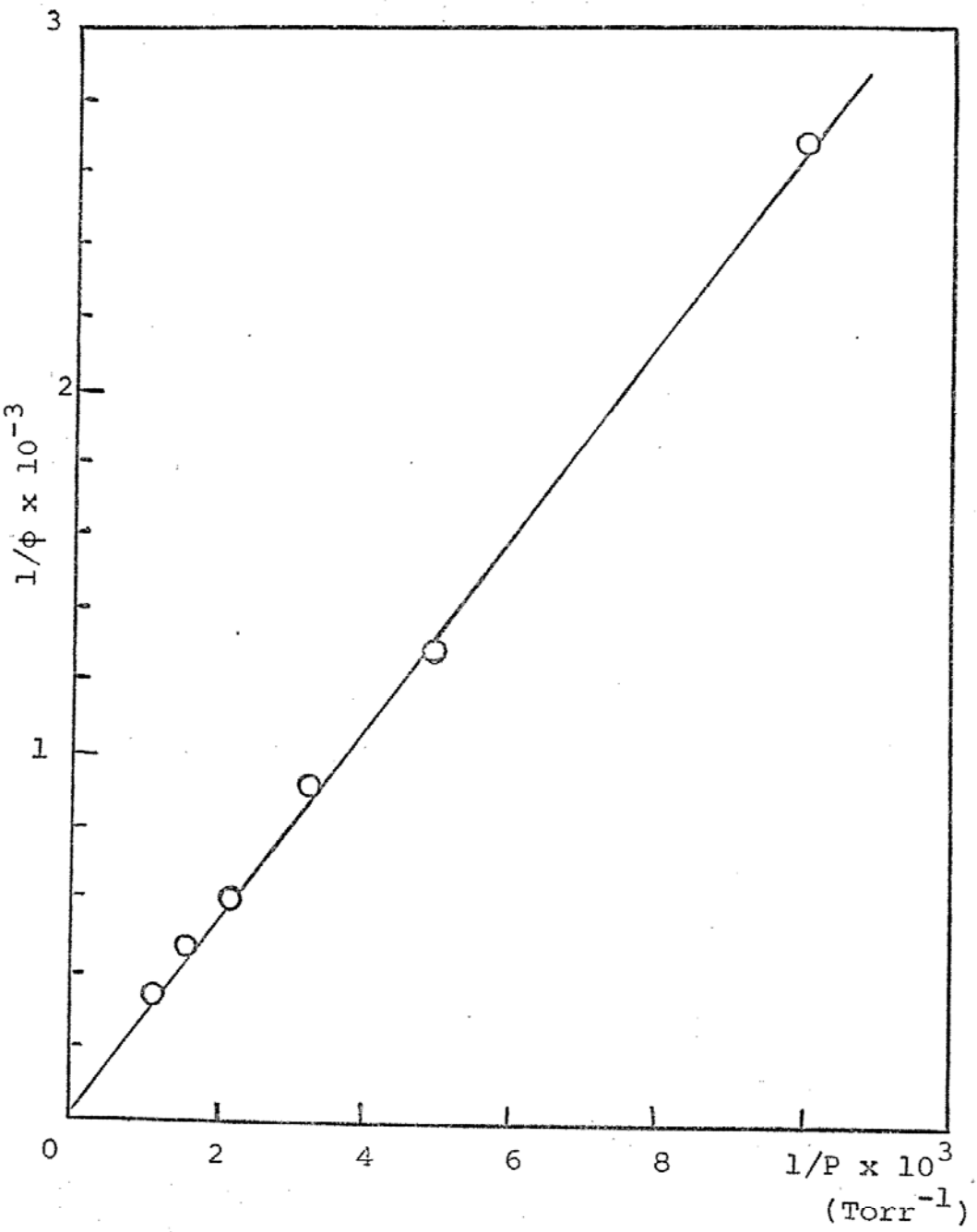


Fig. 4. Reciprocal of Quantum Yield of Photo-product versus Reciprocal of Nitrogen Pressure

there would be a number of upper vibrational levels involved.

(b) The processes following absorption were: collision with nitrogen molecules to cause vibrational deactivation, chemical conversion to high vibrational levels of isomer, which might convert to the ground state product or back to the parent PFB molecule.

Applying the steady state treatment to the above mechanism gives the expression for quantum yield of product formation:

$$\phi_A = \frac{k_3}{k_2[M]+k_3} \cdot \frac{k_6[M]}{k_5+k_6[M]} \qquad (7)$$

where the square bracket represents the concentration in moles per liter. Since the quantum yields were extremely low, it may be assumed that $k_5 \gg k_6[M]$. The above equation then can be simplified to:

$$\frac{1}{\phi_A} = \frac{k_2 k_5}{k_3 k_6} + \frac{k_5}{k_6} \cdot \frac{1}{[M]} \qquad (8)$$

From equation (8), it is apparent that a plot of $1/\phi_A$ versus $1/[M]$ should yield a straight line with a positive slope. This is confirmed by Fig. 4.

The measured quantum yield of photo product formation appears to be too small to account for the apparent quantum deficit. As the assumption given before indicates, step 5 in the mechanism had a rate high enough to cause a low quantum yield. As in benzene, the photo products initially formed may be subject to further photolysis and sensitized rearomatization, and this is probably true for the back isomerization of PFB.

The large quantum deficit indicates that besides chemical relaxation, radiationless transitions may be one of the important processes. In general, radiationless transitions should become less rapid as the energy separation of the electronic terms of the two states involved increases. Thus, internal conversion from the first excited singlet state of a molecule to the ground state and intersystem crossing from the lowest triplet to the ground state should be relatively slow processes, since the energy separation in these cases is usually relatively large. Isomerization has been proposed[3] as an alternative mode of interal conversion ($S_1 \rightarrow S_0$). The excited molecule is considered to undergo internal conversion to a vibrationally excited state of a ground state isomer, which after vibrational relaxation reverts to the normal ground state. With this hypothesis it can be seen that the isomer provides an additional ladder down which the

molecule may dissipate energy by collision and thus removes the difficulty of the direct conversion of the entire electronic energy into vibrational energy. If this is the case, from the above mechanism the excited $^1PFB_v^I$ molecule will return to its ground state via steps 3 and 5. From Fig. 4, the ratio of k_5/k_6 and the ratio of k_2/k_3 are found to be 14.2 and 1.22, respectively.

V. CONCLUSION

The sum of the quantum yields of the two photo isomers from PFB was 0.01. This result is too low to explain the large quantum deficit of PFB excited at 253.7 nm region. The large quantum deficit is a common phenomenon of highly substituted benzene. The reason could be the closely spaced energy levels due to the "heavy" atom substituent.

REFERENCES

(1) D. Phillips, *J. Photochem.* **1**, 97 (1972/73).
(2) S. L. Lem, G. P. Semeluk and I. Unger, *Can. J. Chem.* **49**, 1567 (1971).
(3) A. A. Lamola, G. S. Hammond and F. B. Mallory, *Photochem. Photobiol.*, **4**, 259 (1965).
(4) D. Phillips, J. Lemaire, C. S. Burton and W. A. Noyes, Jr., "*Advances in Photochemistry*", Vol. 5, (1968) pp. 329.
(5) L. Kaplan and K. E. Wilzbach, *J. Am. Chem. Soc.*, **90**, 3291 (1968).
(6) L. Kaplan, S. P. Walch and K. E. Wilzbach, *J. Am. Chem. Soc.*, **90** 5646 (1968).
(7) J. M. Blair and D. Bruce-Smith, *Proc. Chem. Soc.*, 287, (1957).
(8) H. J. F. Angus, J. M. Blair and D. Bruce-Smith, *J. Chem. Soc.*, 2003, (1960).
(9) G. P. Semeluk and R. D. S. Stevens, *Chem. Commun.*, **24**, 1720 (1970).
(10) I. Haller, *J. Am. Chem. Soc.*, 88, 2070 (1966).
(11) C. Camaggi and F. Gozzo, *J. Chem. Soc.*, (C), 489 (1969).
(12) E. Ratajczak, *Rocz. Chem.*, 44, 447 (1970).
(13) P. Cadman, E. Ratajczak and A. F. Trotman-Dickenson, *J. Chem. Soc.*, (A), 2109 (1970).
(14) C. G. Hatchard and C. A. Parker, *Proc. Roy. Soc.*, (London), **A235**, 518 (1956).

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